Bul. Stiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XIV(1998), Nr. 2, 179- 184

A CONTRACTIVE METHOD FOR THE PROOF OF PICARD'S THEOREM

Cristinel MORTICI

Abstract. *In this paper we give another way to establish the unique locally solvability of the Cauchy problem*

$$(1)\qquad \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases},$$

asking the same conditions as in Picard's theorem, namely continuity and lipschizianity with respect the second argument for f. We prove that the differentiation operator $Dy = y'$ *defined between some two Banach spaces is inversable and we write (1) as a fixed point problem:*

$$v(x) = f(x, D^{-1}v(x)),$$

with $v = Dy \Leftrightarrow y = D^{-1}v$ *which is studied using the contraction principle of Banach. In some cases the corresponding approximation sequence is easier to calculate than the sequence from Picard's theorem.*

Let $\Delta = \{(x,y) \in \mathbb{R}^2 \mid |x - x_0| \le a,\ |y - y_0| \le b\}$ be a rectangle, $f : \Delta \to \mathbb{R}$ be a continuous function satisfying Lipschiz condition

$$|f(x,y) - f(x,z)| \le L\,|y - z|,$$

for each (x,y), $(x,z) \in \Delta$ and some L >0.

Let us choose $0 < \varepsilon < \min\left\{a, \frac{b}{M}\right\}$, where $M = \max_{(x,y)\in\Delta} |f(x,y)|$ and denote

$I = (x_0 - \varepsilon, x_0 + \varepsilon)$.

For the beginning we assume that $y_0 = 0$ without losing of generality, as we can see later. Let us consider the Cauchy problem

(2) $$\begin{cases} y' = f(x,y) \\ y(x_0) = 0 \end{cases}.$$

The space defined by $W := \{ y \in \overline{C}^1(I) \mid y(x_0) = 0 \}$ is a closed subspace of $\overline{C}^1(I)$, hence W endowed with $\| \cdot \|_{\overline{C}^1(I)}$ is a Banach space. First we give the following:

Lemma. *The operator* $D : W \to C(\bar{I})$, $Dy = y'$ *is linear, one-to-one and onto. Its inverse* $D^{-1} : C(\bar{I}) \to W$ *is linear, continuous and*

(3) $$\| D^{-1} v \|_{\overline{C}(I)} \le \varepsilon \| v \|_{\overline{C}(I)}, \ \forall \ v \in C(\bar{I}).$$

Proof. Let $y_1, y_2 \in W$ be such that $Dy_1 = Dy_2 \to y_1' = y_2' \to y_1 - y_2$ is constant. But $y_1(x_0) = y_2(x_0) = 0$, that is $y_1 = y_2$.

For every $v \in C(\bar{I})$, there exists $y \in W$, $y(x) := \int_{x_0}^{x} v(t)\, dt$ such that $Dy = v$. Moreover

$$| D^{-1} v(x) | = \left| \int_{x_0}^{x} v(t)\, dt \right| \le | x - x_0 | \cdot \sup_{t \in I} | v(t) | \le \varepsilon \| v \|_{\overline{C}^1(I)}. \quad \square$$

The Cauchy problem (2) can be equivalently written as

(4) $$Dy(x) = f(x, y(x))$$

with $y \in W$. If we put $Dy = v \in C(\bar{I}) \Leftrightarrow y = D^{-1}v$, we have

(5) $$v(x) = f(x, D^{-1}v(x)).$$

Let us consider the operator $S : \overline{B}_M(0) \to \overline{B}_M(0)$,

(6) $$Sv(x) := f(x, D^{-1}v(x)),$$

where $\overline{B}_M(0) = \{ v \in C(\bar{I}) \mid \| v \|_{C(\bar{I})} \le M \}$. S is well defined because f and D^{-1} are continuous. Moreover, if $\| v \|_{C(\bar{I})} \le M$, then

$$| D^{-1}v(x) | \le \varepsilon \| v \|_{C(\bar{I})} \le \frac{b}{M} \cdot M = b,$$

thus $(x, D^{-1}v(x)) \in \Delta$, $(\forall)\ x \in I$. Now we can see that (5) is equivalent with the following fixed problem

(7) $$v(x) = Sv(x).$$

We shall prove that S is a contraction. Indeed, for $v_1, v_2 \in \overline{B}_M(0)$, we have

$$| Sv_1(x) - Sv_2(x) | = | f(x, D^{-1}v_1(x)) - f(x, D^{-1}v_2(x)) | \le$$

$$\le L \cdot | D^{-1}v_1(x) - D^{-1}v_2(x) | = L \cdot | D^{-1}(v_1(x) - v_2(x)) | \le L\varepsilon \| v_1 - v_2 \|.$$

We obtained

(8) $$\| Sv_1 - Sv_2 \| \le c \| v_1 - v_2 \|, \quad \forall\ v_1, v_2 \in \overline{B}_M(0),$$

with $c := L\varepsilon < 1$ if we take $\varepsilon < \frac{1}{L}$.

From contraction principle of Banach it results that S has an unique fixed point denoted by $v \in \overline{B}_M(0) \subset C(\bar{I})$:

(9) $$v(x) = f(x, D^{-1}v(x));$$

or

(10) $$y'(x) = f(x, y(x)),$$

with $y = D^{-1} v \in W$. Thus $y : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ is the unique solution of the Cauchy problem (2). □

Now we consider the general case when $y(x_0) = y_0$:

(11) $$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}.$$

If we denote $z := y - y_0$ then z satisfy the following Cauchy problem

(12) $$\begin{cases} z' = g(x, z) \\ y(x_0) = 0 \end{cases},$$

where $g(x, z) := f(x, z + x_0)$. Obviously, the problem (12) has an unique solution as we have proved above, because g has the same properties together with f. Also (11) has (locally) an unique solution.

A NUMERICAL EXAMPLE.

Let us consider the Cauchy problem

(13) $$\begin{cases} y' = x^2 + y \\ y(0) = 0 \end{cases},$$

which is a linear differential equation having the unique solution

(14) $$y(x) = 2e^x - x^2 - 2x - 2, \quad x \in \mathbb{R}.$$

In this case $f(x, y) = x^2 + y$ and $|f(x, y) - f(x, z)| = |y - z|$, that is lipschizianity with respect the second argument. The operator S is now defined by

(15) $$Sv(x) = x^2 + \int_0^x v(t)\, dt.$$

Using the above theoretical results, we obtain that (13) has an unique solution $y = D^{-1} v$, where v is the unique fixed point of S. Moreover, v is the limit of the sequence $(v_n)_{n\in N}$ defined recursively by

(16)
$$v_{n+1}(x) = x^2 + \int_0^n v_n(t)\,dt ,$$

where v_0 is arbitrary chosen. If we take $v_0 = 0$, then

$$v_1(x) = x^2 , \quad v_2(x) = x^2 + \int_0^x t^2\,dt = x^2 + \frac{x^3}{3} ,$$

$$v_3(x) = x^2 + \int_0^x \left(t^2 + \frac{t^3}{3} \right) dt = x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot 4} .$$

It is easy to see that

(17)
$$v_n(x) = x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot 4} + \dots + \frac{x^{n+1}}{3\cdot 4\cdot \dots \cdot (n+1)} , \quad n \geq 2 ,$$

or

(18)
$$v_n(x) = 2\cdot \sum_{k=0}^{n-1} \frac{x^k}{k!} - 2x - 2 .$$

For $n \to \infty$ we obtain

(19)
$$v(x) = 2e^x - 2x - 2$$

and the solution of (13) is $y = D^{-1} v$, namely

$$y(x) = \int_0^x v(t)\,dt = 2e^x - x^2 - 2x - 2 . \quad \square$$

Remark. The successive approximation sequence from Picard's theorem is given by

(20) $$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_n(s))\, ds\ ,\quad n > 0\ .$$

In some cases the integral from (20) is more difficult to calculate than the integral from our method:

(21) $$v_{n-1}(x) = f\left(x\, , \int_{x_0}^{x} v_n(t)\, dt \right)\ ,$$

because the integral sign in (21) appears only in the second argument of f.

Received 30.03.1998

Cristinel MORTICI
"Ovidius" University
Departmet of Mathematics
Bd. Mamaia 124, 8700 Constanţa,
ROMANIA
Email: cmortici @ ovidius.ct.ro