Bul. Stiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XIV(1998), Nr. 2, 185-190

ZEROS SET FOR THE SOLUTIONS OF THE DIFFERENTIAL-FUNCTIONAL EQUATIONS

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Abstract. In this paper we give some properties for the zeros of the solutions of a functional-differential equation.

Introduction

The differential-functional equations have been studied in many monographs and in many papers.

The linear equations are a special class of differential-functional equations. For results in this field we quote here [1]-[4].

In this paper we consider a first order linear differential equation with linear deviating of the argument. We give some properties for the zeros of the solutions of this equation and we study the properties for regularly too.

The properties for the zeros of the solutions

We consider the following linear functional-differential equation

$$(1) \qquad y'(x) + p(x)y(x) + q(x)y(\lambda x) = 0, \quad x \in [0, b], \quad 0 < \lambda < 1,$$

where $p, q \in C[0,b]$ and $q(x) \neq 0$ for all $x \in [0,b]$.

We have

Theorem 1. *Let $y \neq 0$ be a solution of the equation (1). We suppose that $x_0 > 0$ is the first zero of this solution. Then x_0 can't be a multiple zero for y.*

Proof. We suppose that x_0 is a double zero for y. Then $y(x_0) = 0$, $y'(x_0) = 0$. From the equation (1) we obtain that $q(x_0)y(\lambda x_0) = 0$. Because $q(x_0) \neq 0$, on results that $y(\lambda x_0) = 0$, which is a contradiction. Therefore x_0 can't be a double zero.

Theorem 2. *Let $y \in C^1[0,b]$ be a solution of the equation (1). If $p, q \in C^1[0,b]$, then $y \in C^2[0,b]$.*

Proof. From (1) we obtain that $y' \in C^1[0,b]$, that is $y \in C^2[0,b]$.

Theorem 3. *If $p, q \in C^k[0,b]$, $k \geq 1$, and y is a solution of the equation (1), then $y \in C^{k+1}[0,b]$.*

Proof. We use the Theorem 2 and the induction on k.

Theorem 4. *Let $y \neq 0$ be a solution of the equation (1). We suppose that $p, q \in C^1[0,b]$. If x_1 is a double zero for y then λx_1 is a simple zero for y.*

Proof. Let x_0 be the first zero of y and $x_1 > x_0$ such that $y(x_1) = 0$, $y'(x_1) = 0$ and $y''(x_1) \neq 0$. From the equation (1) we obtain that $q(x_1)y(\lambda x_1) = 0$. Because $q(x_1) \neq 0$, on results that $y(\lambda x_1) = 0$. But λx_1 can't be a double zero for (1). From the equation (1), by differentiating, we obtain

$$y''(x) + p'(x)y(x) + p(x)y'(x) + q'(x)y(\lambda x) + \lambda q(x)y'(\lambda x) = 0. \tag{2}$$

We suppose that λx_1 is a multiple zero for y. Then $y(\lambda x_1) = 0$, $y'(\lambda x_1) = 0$. Let us consider $x = x_1$ in (2). On results $y''(x_1) = 0$, which is a contradiction.

Theorem 5. *We suppose that $p, q \in C^2[0,b]$. Let $y \neq 0$ be a solution of the equation (1). If x_1 is a three order multiple zero for y, then λx_1 is a double zero for y.*

Proof. Let x_0 be the first zero for y and $x_1 > x_0$ such that $y(x_1) = 0$, $y'(x_1) = 0$, $y''(x_1) = 0$ and $y'''(x_1) \neq 0$. Then we have that $y(\lambda x_1) = 0$ (Theorem 4).

We take $x = x_1$ in (2) and we obtain that $\lambda q(x_1)y'(\lambda x_1) = 0$, that is $y'(\lambda x_1) = 0$. Let us prove that $y''(\lambda x_1) = 0$. From the relationship (2), by differentiating, on results the following equality:

$$y'''(x) + p''(x)y(x) + 2p'(x)y'(x) + p(x)y''(x) + q''(x)y(\lambda x) + \\ +2\lambda q'(x)y'(\lambda x) + \lambda^2 q(x)y''(\lambda x) = 0. \tag{3}$$

We take $x = x_1$ in (3) and we obtain $y'''(x_1) = -\lambda^2 q(x_1)y''(\lambda x_1)$. But $y'''(x_1) \neq 0$ and $q(x_1) \neq 0$. So we have that $y''(\lambda x_1) \neq 0$.

Theorem 6. *We suppose that $p, q \in C^{n-1}[0, b]$. Let $y \neq 0$ be a solution of the equation (1). If x_1 is a n order multiple zero for y, then λx_1 is a $n-1$ order multiple zero for y.*

Proof. We use the induction on n. Let x_0 be the first zero for y and $x_1 > x_0$. We consider the following proposition:

P_n: "If x_1 is a n order multiple zero for y, then λx_1 is a $n-1$ order multiple zero for y".

For $n = 2$ and $n = 3$ this proposition is true (Theorem 4 and Theorem 5).

We suppose that the proposition P_{n-1} is true.

We proof that P_{n-1} implies P_n.

Let x_1 be a n order multiple zero for y. Then x_1 is a $n-1$ order multiple zero for y, and because of P_{n-1} we have that λx_1 is a $n-2$ order multiple zero for y. So we have

$$y(\lambda x_1) = 0,\ y'(\lambda x_1) = 0, \dots, y^{(n-3)}(\lambda x_1) = 0.$$

By differentiating $n-2$ times the relationship (1), we obtain

$$y^{(n-1)}(x) + \sum_{k=0}^{n-2} C_{n-2}^k (p(x))^{(n-2-k)} y^{(k)}(x) + \sum_{k=0}^{n-2} C_{n-2}^k (q(x))^{(n-2-k)} \lambda^k (y^{(k)}(\lambda x)) = 0. \tag{4}$$

We take $x = x_1$. But

$$y(x_1) = 0,\ y'(x_1) = 0, \dots, y^{(n-1)}(x_1) = 0$$

and

$$y(\lambda x_1) = 0,\ y'(\lambda x_1) = 0, \dots, y^{(n-3)}(\lambda x_1) = 0.$$

So we have

$$C_{n-2}^{n-2} q(x_1) \lambda^{n-2} y^{(n-2)}(\lambda x_1) = 0.$$

Thus $y^{(n-2)}(\lambda x_1) = 0$.

By differentiating $n-1$ times the relationship (1), we obtain

$$y^{(n)}(x) + \sum_{k=0}^{n-1} C_{n-1}^{k} (p(x))^{(n-1-k)} y^{(k)}(x) + \sum_{k=0}^{n-1} C_{n-1}^{k} (q(x))^{(n-1-k)} \lambda^{k} (y^{(k)}(\lambda x)) = 0. \tag{5}$$

We take $x = x_1$ in (5). But

$$y(x_1) = 0,\ y'(x_1) = 0, \ldots, y^{(n-1)}(x_1) = 0,\ y^{(n)}(x_1) \neq 0$$

and

$$y(\lambda x_1) = 0,\ y'(\lambda x_1) = 0, \ldots, y^{(n-2)}(\lambda x_1) = 0.$$

So we have

$$y^{(n)}(x_1) + C_{n-1}^{n-1} (q(x_1)) \lambda^{n-1} y^{(n-1)}(\lambda x_1) = 0.$$

Because of $y^{(n)}(x_1) \neq 0$ and $q(x_1) \neq 0$, on results that $y^{(n-1)}(\lambda x_1) \neq 0$.

So we have that P_{n-1} implies P_n.

By applying the induction on n we have that P_n is true for all $n \in \mathbf{N}$, $n \geq 2$.

Remark 1. If $y(0) = 0$ then, from the unicity of the Cauchy problem for the equation (1), we have that $y = 0$.

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Received 8.07.1998