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A SPACE LOSING THE NORMALITY WITH ONE

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Summary. An example of a normal topological space that loses the normality with removing one point gives a simple example of non completely normal space.

1. Introduction

We say that a normal topological space (X, ρ) it completely normal if every subspace (Y, ρ) of (X, ρ) is normal (open sets in (Y, ρ) are of the form $Y \cap U$ for sets U open in (X, ρ).) Well known example of normal space that is not completely normal is the space I^I (see , p. 125). We will show an example of a space that is normal due to the fact, that just one point controls the richness of the closed sets. Removing this point the normality disappears.

2. Example

We will construct our example using

Rational sequence topology Let A be the set of real numbers and for each irrational x we choose a sequence $\{x_n\}$ of rationals converging to it in the Euclidean topology. The rational sequence topology α on A is then defined by declaring each rational open, and selecting the sets $U_n(x) = \cup_{i=n}^{\infty}\{x_i\} \cup \{x\}$ as a basis for the irrational point x.

Then (i) (A, α) is completely regular,

(ii) (A, α) is not normal.

Proof See , p. 87, p. 210. □

Using the rational sequence topological space (A, α) we construct the following

Let $B = A \cup \{b\}$, $b \notin A$. We define β topology on B by taking as a subbasis for a topology all α open subsets of A and sets $\{b\} \cup V$, where V contain all but finitely many irrationals of A and all but countably many rationals of A. Then (i) (B, β) is normal, (ii) (B, β) is not completely normal. Proof (i) Let F and G are disjoint β closed sets. Then at least one of β open sets $B \setminus F$ and $B \setminus G$ contains b. Let $b \in B \setminus F$. The β closed set F contains at most finitely many irrationals $\{c_i\}_{i=1}^{n}$ and at most countably many rationals $\{d_i\}_{i=1}^{\infty}$. Given i the sets $\{c_i\}$ and $G \cap A$ are α closed in a regular space A, hence there exist disjoint α open sets U_i and V_i , such that $c_i \in U_i$, $G \cap A \subset V_i$. We can choose U_i in the form $U_i = \cup_{j=n}^{\infty}\{x_j\} \cup \{c_i\}$ where $\{x_j\}$ is a sequence of rationals converging to $\{c_i\}$ in the Euclidean topology (due to basis of topology α in irrational c_i. U_i is clearly α open in all rationals). Now let $U = \cup_{i=1}^{n} U_i \cup F$. Clearly U is β open (irrational c_i are covered by a β open U_i, rational points of F and U_i are β open). Let V is β - interior of $B \setminus U$. We see that $b \in V$. All points of $G \cap A$ are covered by β open set $\cap_{i=1}^{n} V_i \subset (B \setminus U)$. We have disjoint β open U and V such that $F \subset U$, $G \subset V$. B is normal. (ii) we see that the subset A of B with the induced topology is just topological space (A, α), which is not normal.

Remark 1 *The point $b \in B$ is the 'boss' controling rationals and irrationals in B. Removing the boss the normality disappears ;-)*

References.

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