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ASYMTOTIC BEHAVIOR OF THE NONOSCILLATORY SOLUTION OF THE N-TH ORDER DIFERENTIAL EQUATION WITH DELAY DEPENDING ON THE UNKNOW FUNCTION

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Abstract

The asymptotic behavior of the solutions of the differential equations
$(r_{n-1}(t)(r_{n-2}(t)(\ldots(r_1(t)x'(t))'\ldots)')')' + f(t, x(t), x(\Delta(t, x(t)))) = 0$
is considered.

We consider the n-th order differential equation of the form

$$L_n x(t) + f(t, x(t), x(\Delta(t, x(t)))) = 0 \quad (1)$$

where

$$L_1 x(t) = r_1(t)x'(t)$$
$$L_i x(t) = r_i(t)(L_{i-1}x(t))', \quad \text{for } i = 2,\ldots,n-1, \quad (2)$$
$$L_n x(t) = (L_{n-1}x(t))'.$$

Troughout the paper we shall assume that :

H1. $r_i \in C(R_+, R_+)$ and $\int^{\infty} dt/r_i(t) = \infty$ for $i = 1,2,\ldots,n-1$

H2. $f \in C(R_+ \times R^2, R)$, $f(t, u,v)$ is nondescreasing function in u and v for each fixed t.

H3. $u f(t, u, v) > 0$ for $u.v > 0$ and t arbitrary

H4. $\Delta \in C(R_+ \times R, R)$

H5. There exist a function $\Delta_*(t) \in C(R_+, R)$ and $T \in R_+$ such that $\lim_{t\to\infty} \Delta_*(t) = \infty$ and $\Delta_*(t) \le \Delta(t, x)$ for $t \ge T$,

H6. There exist a function $\Delta^*(t) \in C(R_+,R)$ and $T \in R_+$ such that $\Delta^*(t)$ is nondecreasing function for $t \ge T$ and $\Delta(t,x) \le \Delta^*(t) \le t$ for $t \ge T$, $x \in R$

By a solution of equation (1) is meant a fuction x(t), such that $L_i x(t)$, $1 \le i \le n$ exist and are continuous on $[T,\infty)$ and x(t) satisfies (1). We restrict our considerations to those solutions of (1) which exist on some ray $[T_x,\infty)$ and satisfy

$$\sup\{(y(t)): t_1 \le t < \infty\} > 0 \quad \text{for any } t_1 \in [T_x,\infty).$$

Define $T_{-1} = \inf\{\Delta(t,x) : t \ge T, x \in R\}$.

Lemma 1. Let x(t) be a nonoscillatory solution of equation (1). Then there exist an integer l, $0 \le l \le n$ and $t_1 \ge t_0$ with n+l odd such that

$$x(t) L_i x(t) > 0, \quad 1 \le i \le l, \quad (3)$$

$(-1)^{i-1} x(t) L_i x(t) > 0 \quad , \quad 1 \le i \le n$
for all $t \ge t_1$,

$$\lim_{t\to\infty} | L_i x(t) | = \infty \quad \text{for } i = 1, ..., l-2,$$

$$\lim_{t\to\infty} L_{l-1} x(t) \neq 0 \quad , \quad \lim_{t\to\infty} L_l x(t) \quad \text{is own}$$

and

$$\lim_{t\to\infty} L_j x(t) = 0 \quad \text{for } j = l+1, ..., n-1.$$

Lemma generalizes a well- known lemma of Kiguradze and can be prooved similarly.

A function x(t) satisfying (3) is said to be a function of degree l (see [2]). The set of all nonoscillatory solutions of degree l of equation (1) is denoted by N_l.

If

$$N^1_l = \{x \in N_l : \lim_{t\to\infty} L_l x(t) \neq 0\},$$

$$N^0_l = \{x \in N_l : \lim_{t\to\infty} L_l x(t) = 0\},$$

then $N_l = N^1_l \cup N^0_l$,

$N_0 = N_2 = ... = N_{2k} = \phi$ for n even

and

$N_1 = N_3 = ... = N_{2k+1} = \phi$ for n odd .

We shall use the following notation

$$\phi_{k,T} (r_1, ..., r_j ; t) = \int_T^t 1/r_1(s_1) \int_T^{s_1} 1/r_2(s_2) \; ... \int_T^{s_{j-1}} k/r_j(s_j) \; ds_j ... ds_2 ds_1$$

$\phi_k \; (r_1, ..., r_j ; t) = \phi_{k,0} (r_1, ..., r_j ; t)$, for $j = 1,2, ..., n-1$.

Using H_1 we have that

$$\lim_{t\to\infty} | \phi_{k,T} (r_1, ..., r_j ; t) | = \infty \qquad \text{for all } k \neq 0,$$

$$| \phi_{k,T} (r_1, ..., r_j ; t) | > | \phi_{l,T} (r_1, ..., r_j ; t) | \quad \text{for all}_{\,kl>0,} \; |k| > |l|$$

$$\phi_{k,T} (r_1, ..., r_j ; t) = 0$$

Lemma 2. Let x(t) be a nonoscillatory solution of equation (1) degree $l \ge 1$. Then x(t) possesses one of the following properties:

$$\lim_{t\to\infty} x(t) / \phi_k (r_1, ..., r_l ; t) = c \neq 0 \quad ; \qquad \text{(P1)}$$

$$\lim_{t\to\infty} x(t) / \phi_k (r_1, ..., r_l ; t) = 0 \quad , \lim_{t\to\infty} |x(t)| = \infty ; \qquad \text{(P2)}$$

$$\lim_{t\to\infty} x(t) / \phi_k (r_1, ..., r_l ; t) = 0 \quad , \lim_{t\to\infty} |x(t)| = c_1 . \qquad \text{(P3)}$$

Proof. Let x(t) be a nonoscillatory solution of equation (1) degree l. Using L'Hospital rule we have

$$\lim_{t\to\infty} x(t) / \phi_k(r_1, \ldots, r_l ; t) = \lim_{t\to\infty} L_l x(t)$$

and from the Lemma 1 it follows that Lemma 2 is true.

Remark 1. It is obviously that x(t) possesses P1 (P2 or P3) if and only if $x(t)\in N^1_l$ $(x(t)\in N^0_l)$.

Theorem 1. Let equation (1) has a nonoscillatory solution x(t) degree l possesses property (P1). Then

$$\int^{\infty} |f(t, c\,\phi_k(r_1, \ldots, r_l ; t), c\,\phi_k(r_1, \ldots, r_l ;\ \Delta_*(t)))|\, dt < \infty \qquad (6)$$

for some constants $k\neq 0$ and $c>0$.

Proof. Let x(t) be a nonoscillatory solutions of equation (1) degree l for which

$$\lim_{t\to\infty} x(t) / \phi_k(r_1, \ldots, r_l ; t) = c_1 \neq 0.$$

Without loss of generality, we suppose that $c_1>0$ (the proof for $c_1<0$ is similar). Then there exists $c > 0$ and $t_1 \geq T \geq 0$ such that

$$x(t)\geq c\phi_k(r_1, \ldots, r_l ; t) \ , \qquad (7)$$
$$x(\Delta(t,x(t))) \geq c\,\phi_n(r_1, \ldots, r_l ;\ \Delta_*(t)) \ , \ t\geq t_1$$ (using tts and (4))

Integrating (1) from t to ∞ and using properties of the solution x(t) we obtain

$$\int_t^{\infty} f(s,x(s), x(\Delta(s,x(s))))\, ds < \infty \ .$$

From the last inequality in view of (7) we obtain

$$\int_t^{\infty} f(s, c\,\phi_k(r_1, \ldots, r_l ; s), c\,\phi_k(r_1, \ldots, r_l ;\ \Delta_*(s)))\, ds < \infty$$

which implies (6). This completes the proof.

Theorem 2. Suppose that for each fixed $k\neq 0$ and $T\geq 0$

$$\lim_{l\to 0, |k|>0} \phi_{l,T}(r_1, \ldots, r_{n-1} ; t) / \phi_{k,T}(r_1, \ldots, r_{n-1} ;\ t) = 0 \qquad (8)$$

uniformly on any interval of the form $[\,T', \infty)$, $T' > T$.

If for some $c>0$ and $k\neq 0$ we have

$$\int^{\infty} |f(t, c\,\phi_k(r_1, \ldots, r_{n-1} ; t), c\,\phi_k(r_1, \ldots, r_{n-1} ;\ \Delta^*(t)))|\, dt < \infty \qquad (9)$$

then equation (1) has a solution degree (n-1) with property (P1).

Proof. Suppose that (9) holds for some $c>0$ and $k'>0$. As (8) holds, there exist l_1, $k>l_1>0$, such that

$$\phi_{l_1}(r_1, \ldots, r_{n-1} ; t) < c\,\phi_{k'}(r_1, \ldots, r_{n-1} ;\ t).$$

We choose m such that $0 < m \leq c/2$, $T>0$, $2m < l_1 < k$ and

$$\int_T^\infty f(t, \phi_{2m}(r_1, \ldots, r_{n-1}; t), \phi_{2m}(r_1, \ldots, r_{n-1}; \Delta^*(t)))\, dt \leq m \quad . \tag{10}$$

We define the set

$X=\{x \in C([T_{-1}, \infty), R) \ : \ x(t)=0 \text{ for } t \in [T_{-1}, T) \quad \text{and}$

$\phi_{m,T}(r_1, \ldots, r_{n-1}; t) \leq x(t) \leq \phi_{2m,T}(r_1, \ldots, r_{n-1}; t) \text{ for } t \geq T\}$,
and the mapping $S : X \to C([T_{-1}, \infty), R)$ by the formula

$$S_{x(t)} = \begin{cases} 0 \ , \ T_{-1} \leq t < T \ , \\ \int_T^t 1/r_1(s_1) \int_T^{s_1} \ldots \int_T^{s_{n-2}} 1/r_{n-1}(s_{n-1})(m + \int_{s_{n-1}}^\infty f(s, x(s), x(\Delta(s, x(s))))\, ds) ds_{n-1} ds_{n-2} \ldots ds_2 ds_1, \ t \geq T. \end{cases} \tag{11}$$

It is standard to verify that all conditions of the Schauder-Tychonoff fixed point theorem are fullfilled and therefore there exists $x \in X$ such that $x(t) = Sx(t)$. Differentiating this integral equation we obtain that x(t) is a solution of equation (1) degree (n-1) with property (P1). This completes the proof.

Theorem 3. Let the condition (8) holds. Then equation (1) has a nonoscillatory solution x(t) degree (n-1) with property (P1) if and only if (9) holds for some constants $c>0$ and $k \neq 0$.

Proof. Theorem 3 follows from Theorem 1 and Theorem 2.

Remark 2. Theorems generalize some results from the paper [1]. Exactly if n=2 we obtain Theorem 5 from the paper [1]. If $\Delta(t,x(t)) = h_1(t)$ we obtain Theorems 1,2,3 from the paper [2] in the special case $\varphi(t) = t$, $m = 1$.

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