Bul. Stiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XIV(1998), Nr. 2, 213-226

TOPOLOGIES COMPATIBLE WITH THE CONNECTIVITY IN NETWORKS

Gabriela CRISTESCU

Abstract. In a 2ε-network over an ε-network in a totally bounded metric space the topological property of connectivity by means of arcs is described. A neccessary and sufficient condition for the existence of a topology on the 2ε-network such as the connectivity in topology should be equivalent with the connectivity by means of arcs is proved. Many more examples of connectivities taken from 2D and 3D image analysis are studied according to the possibility of deriving them from a topology.

Keywords. v-connectivity, topology compatible with the v-connectivity.

1. Introduction.

In 1970, F.Wyse[12], after the development of the main metric concepts used in image analysis, the problem of defining a topology on $\mathbf{Z}^2$ in which the notion of connectivity reduces to the concept of 4-connectivity induced by the distance $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$, for $x = (x_1,x_2) \in \mathbf{Z}^2$ and $y = (y_1,y_2) \in \mathbf{Z}^2$, was discussed for the first time. This problem was solved by F.Wyse[12] and A. Rosenfeld[8] for the distance d_1 and it was discussed for the case of other types of connectivities related to various tessallations by J.M.Chassery and M.I.Chenin[1], J.M.Chassery[2], A. Rosenfeld[10] for the 2D image analysis. Various integer approximations of the Euclidean distance in 2D and 3D are ellaborated [3, 4, 5, 11] together with the topological concepts accompaning them [10] but the problem of the existence of a topology in which the notion of connectivity reduces to the concept of connectivity by means of arcs has not been solved. A descriptive approach for the case of $\mathbf{Z}^2$ together with various tessallations was developped [1].

The aim of the present study is to investigate the existence and the number of the topologies having this type of property for the case of the ε-networks in totally bounded metric spaces. Using the same ideas like in .M.Chassery and M.I.Chenin[1], a necessary and sufficient condition for a topology to generate a connectivity equivalent to the connectivity by means of arcs is derived.

2. Connectivity by means of arcs in networks

Let (X,d) be a metric space and $A \subseteq X$ a nonempty subset of X admitting a finite ε-network in X, for $\varepsilon > 0$. In order to construct the support of the investigations that are necessary for the accomplishment of the purposes of this paper the following basic results will be used:

Lemma 2.1. [7] *Let A be a set in the metric space X. If there is a finite ε-network for A in X then there is in A a finite 2ε-network for A.*

Theorem 2.2. [6] *Let X a metric space and a set $A \subseteq X$. The necessary condition for A to be relatively compact is that for every $\varepsilon > 0$ there is in X a finite ε-network for A. If X is complete then the condition is also sufficient.*

The hypothesis of the lemma 2.1. and of the theorem 2.2. are always fulfilled in the case when A is a rectangle together with its interior (for the 2D-images) or a parallelepiped toghether with its interior (in the case of 3D-scenes).

Let then be (X,d) a totally bounded metric space and for a number $\varepsilon > 0$ let E' be a finite ε-network over X. If E is the 3ε-network over E' that exists according to the lemma 2.1. , then

(2.1) $$V(x) = \{y \mid y \in E, d(y,x) \leq 2\varepsilon\}$$

is denoted for every $x \in E$ and

(2.2) $$\mathcal{V} = \{V(x) \mid x \in E\}$$

The set $\mathcal{V}$ has the following properties:

Proposition 2.3. a) *$x \in V(x)$ for every $x \in E$;* b) *$2 \leq card\ V(x) < +\infty$ for every $x \in E$;* c) *$x \in V(y)$ iff $y \in V(x)$.*

The proof is similar to that of the $\mathbf{Z}^2$-case[1].

All over this paper the following condition will be supposed to be fulfilled in addition:

(2.3) $$V(x) = V(y) \text{ iff } x = y.$$

In the 2D-case both 4-neighbourhoods defined by the distance d_1 and the 8-neighbourhoods defined by the distance $c(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$ are satisfying (2.3). The above mentioned properties are also fulfilled in the case of the triangular tessallations [1] and in the 2ε-networks determined by the hexagonal and octagonal distances [3, 4, 5, 11].

Definition 2.1. *Let x and y be two points of E. A v-path from x to y is a succesion of points from E, $x = x_0, x_1, x_2, \ldots, x_n = y$, such that*

$x_i \in V(x_{i-1})$ for every $i = 1, 2, \ldots, n$.

Definition 2.2. *The points x and y from a set $S \subset E$ are said to be v-connected if there is a v-path from x to y completely contained in S.*

Property 2.4. *The relation of "v-connectivity" in S in an equivalence in S.*

Definition 2.3. *An equivalence class with respect to the relation of "v-connectivity" in S is said to be a v-connected component of S. If S has an unique v-connected component then S is said to be v-connected.*

A theory of arcs and curves becomes possible now, but it is very complex even in the 3D-case [9], and it is not the subject of this study.

Property 2.5. *The sets $\varnothing$ and $\{x\}$ are v-connected, for every $x \in E$.*
Property 2.6. *$\{x,y\}$ is v-connected iff either $x \in V(y)$ or $y \in V(x)$.*

Property 2.7. *For every $x \in E$, $V(x)$ is v-connected.*

Property 2.8. *If A and B are two v-connected components with a nonempty intersection then $A \cup B$ is a v-connected component.*

The proofs are similar to the $\mathbf{Z}^2$-case[1].

Remark 2.1. All the results from this section and from the next one can be obtained more generally, if, for a certain tessallation, V(x) is defined as a set verifying as axioms the properties included in the proposition 2.3 and the condition (2.3). In this paper the particular case of a 2ε-network over an ε-network in a totally bounded metric space is prefered because this is the environment of the field of the image analysis we are dealing with. For the case of 2D-images these results are wellknown [1].

In what follows, two additional conditions will be supposed to be satisfied by the v-connectivity:

(2.4) *E is v-connected;*

(2.5) *For every $x \in E$ the set $E-V(x)$ has a finite number of v-connected components.*

3. Topologies compatible with the v-connectivity.

Let σ be a topology on E. A set $A \subset E$ is said to be connected in σ if there is not a partition of A consisting from two open subsets.

Definition 3.1. *The topology σ is said to be compatible with the v-connectivity if the v-connected subsets in E are the connected subsets in the sense of the topology σ.*

Now, σ is supposed to be a topology compatible with the v-connectivity in E and the properties of such a topology are investigated. For every point $x \in E$ the set D(x) is defined as the intersection of all the open sets (in σ) containing x.

Property 3.1. *For every $x \in E$, the following properties are true:*

a) $D(x) \subseteq V(x)$; b) D(x) is open.

Proof. a) For $x \in E$ the set $P = [\, E - V(x)\,] \cup \{x\}$ is considered. From the property 2.5 it follows that $\{x\}$ is v-connected. (2.5) implies that

$E - V(x)$ has a finite number of v-connected components. Due to the hypothesis of the compatibility of the topology σ with the v-connectivity, the v-connected components are also connected in the topology σ. Therefore, every v-connected component is closed in the topology induced by σ on $[E - V(x)] \cup \{x\}$. The set $E - V(x)$ is a finite union of closed sets in P, therefore it is closed and its complement in P, $\{x\}$, is open in P. It means that there is an open set D in σ such that $\{x\} = D \cap [[\, E - V(x)] - \{x\}]$, involving that $D \subseteq V(x)$ and, due to the definition of D(x), $D(x) \subseteq V(x)$.

b) If $D = D(x)$ then $D(x)$ is open. Contrariwise there is a point $y \in D\text{-}D(x)$ and an open set D' such that $x \in D'$ and $y \notin D'$. The set $D \cap D'$ is open. If $D \cap D' = D(x)$ then $D(x)$ is open. If this is not true then after a finite number of the iterations of the process just described an intersection equal to D(x) is obtained and therefore D(x) is open. □

Property 3.2. *The set* $\{D(x) \mid x \in E\} \cup \varnothing$ *is a basis for the topology* σ.

Proof. Let $D \in \sigma$. It is obvious that there are only the following two possibilities for D: $D = \varnothing$ or $D = \bigcup_{x \in D} D(x)$. Therefore, $\{D(x) \mid x \in E\} \cup \varnothing$ is a basis for the topology σ. □

Property 3.3. *If* $x \in E$ *and* $y \in E$ *then* $x \in D(y)$ *iff* $D(x) \subseteq D(y)$.

The proof is evident due to the definition of D(x).

Property 3.4. *If* $x \in E$ *and* $y \in E$ *then the following propositions are equivalent:* a) $\{x,y\}$ *is a connected set in* σ; b) $\{x,y\}$ *is a v-connected component*; c) $x \in V(y)$; d) $y \in V(x)$; e) $x \in D(y)$ *or* $y \in D(x)$; f) $D(x) \subseteq D(y)$ *or* $D(y) \subseteq D(x)$.

The proof of the property 3.4 is similar to that of its correspondent in the plane[1]. The following properties are also similar to their plane particular case [1]

Property 3.5. *If* $x \in E$ *and* $y \in E$ *then:*

a) $D(x) = D(y)$ *iff* $x = y$;

b) If $x \in D(y)$ *and* $x \neq y$ *then* $y \notin D(x)$.

Property 3.6. *If* $x \in E$ *then for every* $y \in D(x)$ *from* $z \in V(x)\text{-}V(y)$ *it follows that* $z \in D(x)$.

The results will be obtained in what follows are stronger then the similar ones in the plane[1], leading to a necessary and sufficient condition for the existence of a topology in which the connectivity property is equivalent to the v-connectivity.

Lemma 3.7. *Let* $x \in E$ *and* $y \in E$ *such that* $V(x) \cap V(y)$ *is a v-connected set. Then*

$$V(x) = \{x,y\} \cup [V(x) - V(y)] \cup \bigcup_{z \in V(x) - V(y)} [V(x) - V(z)]. \tag{3.1}$$

Proof. It is easy to see that the sets $\{x,y\}$, $V(x) - V(y)$ and

$\bigcup_{z \in V(x) - V(y)} [V(x) - V(z)] - \{y\}$ are a partition of the neighbourhood $V(x)$. □

Lemma 3.8. *Let $x \in E$ and $y \in E$ such that the set $V(x) \cap V(y)$ is v-connected iff either $V(x) \cap V(y) = \varnothing$ or $V(x) \cap V(y)$ is singleton or $V(x) \cap V(y) = \{x,y\}$. Then*

(3.2) $$V(x) = \{x,y\} \cup [V(x) - V(y)].$$

Proof. If the hypothesis is fulfilled then either $V(x)\cap V(y)=\varnothing$ or $V(x)\cap V(y)$ is singleton or $V(x)\cap V(y)=\{x,y\}$. Indeed, if $V(x)\cap V(y)$ is supposed to be both nonempty and not singleton and if a point $z \in V(x)\cap V(y)$, $z \neq x$, $z \neq y$, is supposed to exist then $z \in V(x)$ and $z \in V(y)$, therefore z, x, y is a path from z to y. This implies that $V(x) \cap V(y)$ is a v-connected set, contradiction with the hypothesis. In this case, the sets $\{x,y\}$ and $V(x)$-$V(y)$ from a partition of $V(x)$. □

Theorem 3.9. *The hypothesis that for every two points $x \in E$ and $y \in E$ the set $V(x) \cap V(y)$ is v-connected iff either $V(x) \cap V(y) = \varnothing$ or $V(x) \cap V(y)$ is singleton or $V(x) \cap V(y) = \{x,y\}$ is assumed to be valid. Then if σ is a topology compatible with the v-connectivity then it follows that either*

$D(z) = \{z\}$ or $D(z) = V(z)$ for every $z \in E$.

Proof. The condition $D(z) \neq \{z\}$ is supposed. From the property 3.1 it follows that $D(z) \subseteq V(z)$. Let $y \in D(z)$ with $y \neq z$. The property 3.6 implies that $V(z)$-$V(y) \subset D(z)$, and the lemma 3.8 implies that $V(z)=\{y,z\}\cup[V(z) - V(y)]$. Then $\{y,z\}\subset D(z)$ and the inclusion

$V(z) - V(y) \subset D(z)$ involves that $V(z) \subseteq D(z)$. Therefore $V(z) = D(z)$. □

Theorem 3.10. *For every $x \in E$ and $y \in E$ the set $V(x) \cap V(y)$ is supposed to be v-connected. Then if σ is a topology compatible with the v-connectivity then it follows that either $D(z) = \{z\}$ or $D(z) = V(z)$ for every*

$z \in E$.

Proof. The same method as that used for the proof of the theorem 3.9 leads to the result, but instead of the lemma 3.8, the lemma 3.7 is used. □

Theorem 3.11. *The necessary and sufficient condition for the existence of a topology compatible with the v-connectivity in E is that the set*

$V(x) \cap V(y)$ is v-connected iff either $V(x) \cap V(y) = \varnothing$ or $V(x) \cap V(y)$ is singleton or $V(x) \cap V(y) = \{x,y\}$.

Proof. As a consequence of the theorems 3.9 and 3.10 it follows that either $D(x) = \{x\}$ or $D(x) = V(x)$ for every $x \in E$. If σ is the discrete topology then $D(x)=\{x\}$ for every $x \in E$. Let $y \in V(x)$, $y \neq x$. Then $\{x,y\}$ is v-connected. But from $D(y)=\{y\}$ it follows that $\{x,y\}$ is not connected in σ. Therefore, σ is not the discrete topology, so there is $x \in E$ such that

$D(x) = V(x)$. Let now $y \in V(x)$, $y \neq x$. Then $y \in D(x)$ and $y \neq x$. According to the property 3.5 it follows that $x \notin D(y)$. But the proposition 3.3 implies $x \in V(y)$ and therefore $D(y) \neq V(y)$, so $D(y) = \{y\}$.

Suppose now that there is a point $t \in E$ such that $V(x) \cap V(t) = \{y,z\}$, with $z \neq x$ and $z \neq y$, $z \neq t$, $y \neq x$ and $\{y,z\}$ is v-connected. But $D(z) = \{z\}$ and $D(y) = \{y\}$, involving that $\{y,z\}$ is not connected in σ, being the union of two open separated sets.

For the converse, the hypothesis that for every $x \in E$ and $y \in E$ the set $V(x) \cap V(y)$ is v-connected iff either $V(x) \cap V(y) = \emptyset$ or $V(x) \cap V(y)$ is singleton or $V(x) \cap V(y) = \{x,y\}$ is supposed. A topology compatible with the v-connectivity, σ, will be constructed. From the theorem 3.9 it follows that for every point $x \in E$ the condition $D(x)=\{x\}$ or $D(x) = V(x)$ takes place and, according to the previous remark, there is $x \in E$ such that $D(x) = V(x)$. The topology σ will be constructed by defining the sets $D(x)$ for every $x \in E$.

Let a point $x \in E$. For this point, let $D(x) = V(x)$. Then, according to the previous remark, for every $y \in V(x) - \{x\}$ the condition $D(y) = \{y\}$ takes place. Let $A_1 = \{z \in E - \{x\} \mid V(z) \cap V(x)$ is either singleton or is not v-connected$\}$.

For every $z \in A_1$, $D(z) = V(z)$ is taken. Then for every $t \in V(z)$ it follows that $D(t) = \{t\}$. Let

$A_2 = \{z \in E - A_1 \mid \exists\, y \in A_1, V(z) \cap V(y)$ is either singleton or is not v-connected$\}$.

For every $z \in A_2$, $D(z) = V(z)$ is taken. Then for every $t \in V(z)$ it follows that $D(t) = \{t\}$. The method will be continued until, for every $x \in E$, the smallest open set $D(x)$ will be defined. The topology σ is now that hawing the set $\{D(x) \mid x \in E\}$ as a base. Its compatibility with the v-connectivity will be proved. Evident, $E = \cup \{D(x) \mid x \in E\}$.

Let M be a connected set in σ, let $x \in M$ and let $M(x)$ be the v-connected component containing x. Then $y \in M(x)$ implies $D(y) \cap M \subseteq M(x)$ and, similarly, $y \in M - M(x)$ implies $D(y) \cap (M-M(x)) \subseteq M-M(x)$, therefore both M and $M-M(x)$ are open sets in M. A consequence of the connectivity of M is that $M-M(x)$ must be empty, therefore $M=M(x)$ and this means that M is v-connected.

Conversely, let P be a v-connected set in E and suppose that $P = A \cup B$, with $A \subset E$, $B \subset E$ and $A \cap B = \emptyset$. Let $a \in A$ and $b \in B$ and let $a=a_1, a_2, \ldots, a_n=b$ a v-path in P. Then there is a number k such that $a_k \in A$ and $a_{k+1} \in B$. Then either $a_k \in D(a_{k+1})$ or $a_{k+1} \in D(a_k)$, as a consequence of the property 3.4. This means that either A or B cannot be open, therefore P is connected in σ.□

From the proof of the theorem 3.11 more properties having a very important and practical content can be obtained.

Corollary 3.12. *The discrete topology in E is not compatible with the v-connectivity.*

Corollary 3.13. *If σ is a topology in E that is compatible with the v-connectivity than there is $x \in E$ having the property that $D(x)=V(x)$.*

Corollary 3.14. *If σ is a topology compatible with the v-connectivity in E and if for $x \in E$ the condition $D(x)=V(x)$ takes place than for every $y \in V(x)-\{x\}$ it follows that $D(y)=\{y\}$.*

Corollary 3.15. *The topology σ constructed in the proof of the theorem 3.11 is the unique type of topology compatible with the v-connectivity in E.*

Proof. If the existence of a topology σ' of another type is supposed then in that topology the following condition is valid: there is a natural number k such that $D(z)=V(z)$ for a point $z \in A_k$, but $D(t)=V(t)$ for a point $t \in V(z)$. But this is a contradiction with the corollary 3.14. □

Remark 3.1. In the proofs of the theorems from this paragraph the metrical properties of the 2ε-network have not been used. Only the results of the previous paragraph are involved, but, according to the remark 2.1, they are valid in more general conditions. Therefore, all the results from 3 are true for every tessallation over a set X such that it is possible to associate to every point x of the network that is obtained after the tessallation a set V(x) sattisfying the conditions of the proposition 2.3 together with (2.3).

4. Applications for 2D and 3D tessallations

4.A. The case of the triangular tessallation in 2D

For the case in which the plane $\mathbf{R}^2$ is covered by a pavement consisting of equilateral triangles, every triangle being represented by its center of gravity, E is the collection of these centers (fig.4.1). Then the set $V(x) = \{x, a, b, c\}$ for $x \in E$, defined like in the figure 4.1 sattisfies the proposition 2.3 and (2.3). So, the conditions of the lemma 3.8 and the theorem 3.9 are fulfilled and then, according to the theorem 3.11, there is a topology σ compatible with the v-connectivity.

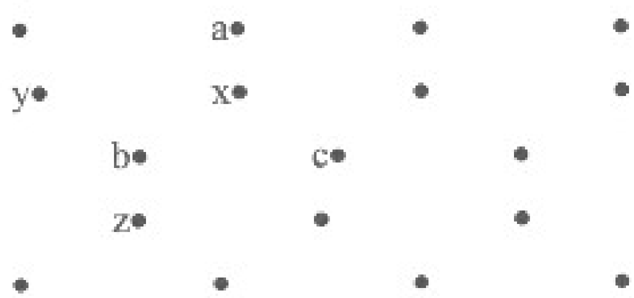

Fig. 4.1.

This topology is constructed like in the proof of the theorem 3.11. In fact, there are two topologies of this type:

1) The topology σ_1 defined by:

D(x)=V(x) if V(x) has the configuration of V(x) from the fig.4.1;

D(x) = {x} contrariwise,

2) the topology σ_2 defined by:

D(x)=V(x) if V(x) has the configuration of V(b) from the fig.4.1;

D(x) = {x} contrariwise.

These results was obtained by J.M.Chassery and M.I.Chenin[1] and J.M.Chassery[2].

4.B. The case of the tetrahedral 3D tessallation

In the case of three dimensions, when the space $\mathbf{R}^3$ is divided into regular tetrahedrons, the set E consists from the centers of gravity of these tetrahedrons. The conditions of the lemma 3.8 and of the theorem 3.9 are satisfied, therefore, according to the theorem 3.11, there is a topology σ compatible with the v-connectivity in E. In fact, in this case there are also two topologies, described in a similar way as in the above mentioned 2D case.

4.C. The case of the city-block distance in 2D

In $\mathbf{R}^2$ the 1/2-net $\mathbf{Z}^2$ is considered, being determined by the city-block $d_1(a,b)=|x_a-x_b| + |y_a-y_b|$, where $a=(x_a, y_a) \in \mathbf{R}^2$ and $b=(x_b, y_b) \in \mathbf{R}^2$. Therefore, $\mathbf{Z}^2$ becomes an 1-net over $\mathbf{Z}^2$, according to the lemma 2.1. In this case, for every $a \in \mathbf{Z}^2$, $V(a)=V_4(a)=\{(x_a-1, y_a), (x_a+1, y_a), (x_a, y_a-1), (x_a, y_a+1), a\}$. The hypothesis of the lemma 3.8 and the theorem 3.9 are fulfilled and then the process described in the proof of the theorem 3.11 leads to the two topologies discussed by A.Rosenfeld[10] and F.Wyse[12].

4.D. The case of the city-block distance in 3D

In the same way as in the 2D case, the set $\mathbf{Z}^3$ ia a 1/2-net over $\mathbf{R}^3$ and an 1-net over $\mathbf{Z}^3$, determined by the city-block distance

$d_1(a,b) = |x_a-x_b| + |y_a-y_b| + |z_a-z_b|$, where $a=(x_a, y_a, z_a) \in \mathbf{R}^3$

and

$b=(x_b, y_b, z_b) \in \mathbf{R}^3$.

This case satisfies the theorem 3.11, generating two topologies σ_1 and σ_2 that are compatible with the v-connectivity. In this case $V(a)=\{a, (x_a-1, y_a, z_a), (x_a+1, y_a, z_a), (x_a, y_a-1, z_a), (x_a, y_a+1, z_a), (x_a, y_a, z_a-1), (x_a, y_a, z_a+1)\}$. Then the basis of the topology σ_1 consists of the sets D(x), $x = (i,j,k) \in \mathbf{Z}^3$, defined by

(4.1) $$D(x) = \begin{cases} V(x) & \text{if} \quad i+j+k \quad \text{is even} \\ \{x\} & \text{if} \quad i+j+k \quad \text{is odd} \end{cases},$$

and the basis of the topology σ_2 is defined by

(4.2) $$D(x) = \begin{cases} \{x\} & \text{if} \quad i+j+k \quad \text{is even} \\ V(x) & \text{if} \quad i+j+k \quad \text{is odd} \end{cases}.$$

4.E. The case of max, hexagonal and octagonal distances in 2D and their extensions in 3D

Both in the case of the 8-connectivity defined by the distance $c(a,b) = \max\{|x_a - x_b|, |y_a - y_b|\}$, for $a=(x_a, y_a) \in \mathbf{R}^2$ and $b=(x_b, y_b) \in \mathbf{R}^2$ and in the case of hexagonal distances [3, 11] and octagonal distances [4, 5], the hypothesis of the theorem 3.9 is not satisfied by sets having special configurations. Therefore, according to the theorems 3.10 and 3.11, in these cases there is not a topology compatible with the v-connectivity. The same result is also valid for the connectivity defined in $\mathbf{Z}^3$ by means of the extensions in 3D of the above mentioned distances. This result was obtained by J.M.Chassery [2] for the case of the 8-connectivity.

A discussion on these special configurations will be developed in what follows.

If the totally bounded metric spaces X reduces to the case of the 2D unit ball determined by the chessboard distance c in the plane then, for $\varepsilon = 1/2$ one can consider

$E=\{(0,0), (-1,0),(-1,-1), (-1,1),(0,1),(0,-1),(1,0),(1,1),(1,-1)\}=V((0,0))$. Denoting these points by letters, one has

g•	f•	e•
h•	a•	d•
i•	b•	c•

Fig. 4.2.

and the result of J.M-Chassery [2] is the following.

Theorem 4.1. [2] *There are four digital topologies in* E *that are compatible with the* v*-connectivity.*

These topologies are generated by the following basis:

(σ_1) D(a) = {a} , D(b) = {a,b,c,d,h,i}, D(c) = {a,c,d}, D(d) = {a,d}
D(e) = {a,d,e}, D(f) = {a,d,e,f,g,h}, D(g) = {a,g,h}, D(h) = {a,h}
D(i) = {a,h,i}

(σ_2) D(a) = {a,b,c,d,e,f,g,h,i}, D(b) ={b,c,d,h,i},D(c)={c,d}, D(d) = {d}

D(e)={c,d},D(f)={d,e,f,g,h}, D(g) = {g,h}, D(h) = {h}, D(i) = {h,i}

The other two topologies are obtained applying a $\pi/2$ -rotation on σ_1 and σ_2, therefore:

(σ_3) D(a) = {a}, D(b) = {a,b}, D(c) = {a,b,c}, D(d) = {a,b,c,d,e,f}
D(e) = {a,e,f}, D(f) = {a,f}, D(g) = {a,f,g}, D(h) = {a,b,f,g,h,i}
D(i) = {a,b,i}

(σ_4) D(a)={a,b,c,d,e,f,g,h}, D(b) = {b}, D(c) = {b,c}, D(d) = {b,c,d,e,f}
D(e)={e,f}, D(f)={f}, D(g) = {f,g}, D(h) = {a,b,f,g,h,i}, D(i)= {b,i}.

According to the theorems 3.10 and 3.11 it is evident that for a set in $\mathbf{Z}^2$ containing a subset similar to that one presented

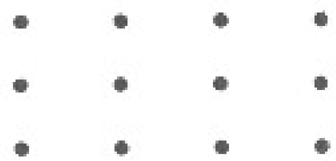

Fig. 4.3

in the fig.4.3 there is not a topology compatible with the v-connectivity. This is a stronger result that the conclusion obtained by J.M.Chassery [2] . The extension for the case of the 3D cube or more is now natural.

In the case of the hexagonal distance[11]

(4.3) $$d_6(x,y) = \max \{ |i-h|, (1/2)(|i-h|+(i-h)) - ([i/2] - [h/2])+k-j, (1/2)(|i-h| + (i-h))+ ([i/2] - [h/2])+j-k\}$$

for $x = (i,j) \in \mathbf{Z}^2$ and $y = (h,k) \in \mathbf{Z}^2$, where [.] means the integer part. For $\varepsilon\geq1$ (therefore $2\varepsilon\geq2$), the ball $V_6(x, 2\varepsilon)$ in $\mathbf{Z}^2$ has either the configuration from the figure 4.4 or that from the figure 4.5[11].

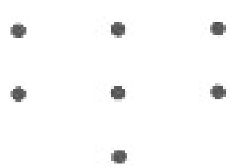

Fig. 4.4

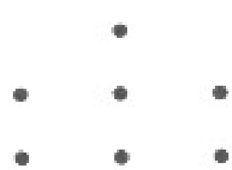

Fig. 4.5

Therefore, the existence of a topology compatible with the v-connectivity on a hexagon as in the figure 4.6 will be discussed.

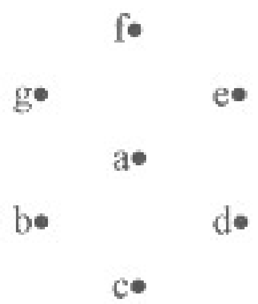

Fig. 4.6

Theorem 4.2 *If* $E = V_6(a)$ *for a point* $a \in \mathbf{Z}^2$, *then there are four topologies compatible with the v-connectivity in E.*

Proof. Let E = {a,b,c,d,e,f,g} as in the figure 4.6. From the identity between the v-connectivity determined by d_6 and the connectivity in the topology we are looking for the following properties are obvious:

(4.4) {x,y} connected $\Leftrightarrow x \in D(y)$ or $y \in D(x)$;

(4.5) $x \in D(y) \Leftrightarrow D(x) \subseteq D(y)$;

(4.6) x and y are 6-neighbours $\Leftrightarrow$ {x,y} is connected $\Leftrightarrow D(x) \subseteq D(y)$ or $D(y) \subseteq D(x)$;

(4.7) x and y are not 6-neighbours $\Leftrightarrow x \notin D(y)$ or $y \notin D(x)$, where $x \in E$, $y \in E$, $x \neq y$.

The existence of a topology compatible with the v-connectivity is supposed. From (4.7) it follows that in this topology the following relations are true:

(4.8) $\{e,d\} \subseteq C\,D(b) \cap C\,D(g)$;

(4.9) $\{e,f\} \subseteq C\,D(b) \cap C\,D(c)$;

(4.10) $\{f,g\} \subseteq C\,D(c) \cap C\,D(d)$,

where C A means the complement of the set A. Because of the v-connectivity of {d,e},(4.6) implies that $D(d) \subseteq D(e)$ or $D(e) \subseteq D(d)$.

Suppose that $D(d) \subseteq D(e)$. Then

$$(4.9)\ \left.\begin{array}{l} c \notin D(e) \\ D(d) \subset D(e) \end{array}\right\} \Rightarrow \left.\begin{array}{l} c \notin D(d) \\ c,\ d\ \text{6-neighbours} \end{array}\right\} \Rightarrow \left.\begin{array}{l} D(d) \subset D(c) \\ c \notin D(d) \end{array}\right\} (4.11)$$

$$(4.10)\ \left.\begin{array}{l} d \notin D(f) \\ D(d) \subset D(e) \end{array}\right\} \Rightarrow \left.\begin{array}{l} D(e) \not\subseteq D(f) \\ e,\ f\ \text{6-neighbours} \end{array}\right\} \Rightarrow \left.\begin{array}{l} D(f) \subset D(e) \\ e \notin D(f) \end{array}\right\} (4.12)$$

$$\left.\begin{array}{l} (4.9) \Rightarrow e \notin D(c) \\ (4.11) \Rightarrow D(d) \subseteq D(c) \end{array}\right\} \Rightarrow \left.\begin{array}{l} e \notin D(d) \\ e,\ d\ \text{6-neighbours} \end{array}\right\} \Rightarrow \left.\begin{array}{l} D(d) \subset D(e) \\ e \notin D(d) \end{array}\right\} (4.13)$$

$$\left.\begin{array}{l} (4.8) \Rightarrow d \notin D(b) \\ (4.11) \Rightarrow D(d) \subseteq D(c) \end{array}\right\} \Rightarrow \left.\begin{array}{l} D(c) \not\subset D(b) \\ b,\ c\ \text{6-neighbours} \end{array}\right\} \Rightarrow \left.\begin{array}{l} D(b) \subset D(c) \\ c \notin D(b) \end{array}\right\} (4.14)$$

$$\left.\begin{array}{l}(4.10)\Rightarrow c\notin D(g)\\(4.14)\Rightarrow D(d)\subseteq D(c)\end{array}\right\}\Rightarrow\left.\begin{array}{l}D(g)\subseteq D(b)\\ b,\ g\ \text{6-neighbours}\end{array}\right\}\Rightarrow\left.\begin{array}{l}D(b)\subset D(g)\\ g\notin D(b)\end{array}\right\}(4.15)$$

$$\left.\begin{array}{l}(4.9)\Rightarrow f\notin D(b)\\(4.15)\Rightarrow D(b)\subset D(g)\end{array}\right\}\Rightarrow\left.\begin{array}{l}D(g)\subseteq D(f)\\ g,\ f\ \text{6-neighbours}\end{array}\right\}\Rightarrow\left.\begin{array}{l}D(f)\subset D(g)\\ g\notin D(f)\end{array}\right\}(4.16)$$

The consequence of all these is that: D(b) = {b,•}, D(c) = {b,c,d,•},

D(d) = {d,•}, D(e) = {d,e,f,•}, D(f) = {f,•}, D(g) = {b,f,g,•} and the situation of the point a remains to be analysed. Suppose that $a\in D(b)$. Then from (4.14) it follows that $a\in D(c)$ and (4.15) implies that $a\in D(g)$. In these conditions $a\notin D(d)$ is supposed. Then D(d)={d} and the sets {d} and {a,g} are a partition of {a,d,g}. But this set is v-connected and therefore it must be connected in the topology. It follows that $a\in D(d)$ and (4.13) implies that $a\in D(e)$. The proof of $a\in D(f)$ is similar and all these results mean that $a\in D(x)$ for every $x\in E$. Therefore $D(a) = \cap\{D(x) \mid x\in E\} = \{a\}$. The first topology σ_1 was now generated by means of its basis:

D(a) = {a}, D(b) = {a,b}, D(c) = {a,b,c,d}, D(d) = {a,d}

D(e) = {a,d,e,f}, D(f) = {a,f}, D(g) = {a,b,f,g}.

Now $a\notin D(b)$ is supposed. If $a\in D(d)$ or $a\in D(f)$, an argument similar to the previous one implies that $a\in D(b)$. But this is a contradiction. Therefore $a\notin D(d)$ and $a\notin D(f)$. If $a\in D(c)$ then the sets D(f)={f} and $\{a,c\}\subset D(c)$ are a partition of {a,c,f}. But {a,c,f} is v-connected and, therefore, it is also connected in the topology, which is a contradiction. Therefore $a\notin D(c)$. A similar argument is used to prove that $a\notin D(g)$ and $a\notin D(e)$. Then, the v-connectivity of E implies that D(a)=E. The topology σ_2 having the basis:

D(a) = E, D(b) = {b}, D(c) = {b,c,d}, D(d) = {d}

D(e) = {d,e,f}, D(f) = {f}, D(g) = {f,g,b}

is obtained.

Now, if $D(e)\subseteq D(d)$ is supposed, by the similar proof the topology σ_3 and σ_4 are obtained as follows:

(σ_3) D(a) = {a}, D(b) = {a,b,c,g}, D(c) = {a,c}, D(d) = {a,c,d,e}

D(e) = {a,e}, D(f) = {a,e,f,g}, D(g) = {a,g};

(σ_4) D(a) = E, D(b) = {b,c,g}, D(c) = {c}, D(d) = {c,d,e}

D(e) = {e}, D(f) = {e,f,g}, D(g) = {g}.□

The following theorem identifies the minimum set having the property that there is not a topology compatible with the v-connectivity defined by means of the distance d_6.

Theorem 4.3 *If $a \in \mathbf{Z}^2$ and $b \in \mathbf{Z}^2$ such that $a \in V_6(b)$, $b \in V_6(a)$ and if $E = V_6(a) \cup V_6(b)$ then there is not a topology in E which is compatible with the v-connectivity.*

Proof In this case E is one of the sets:

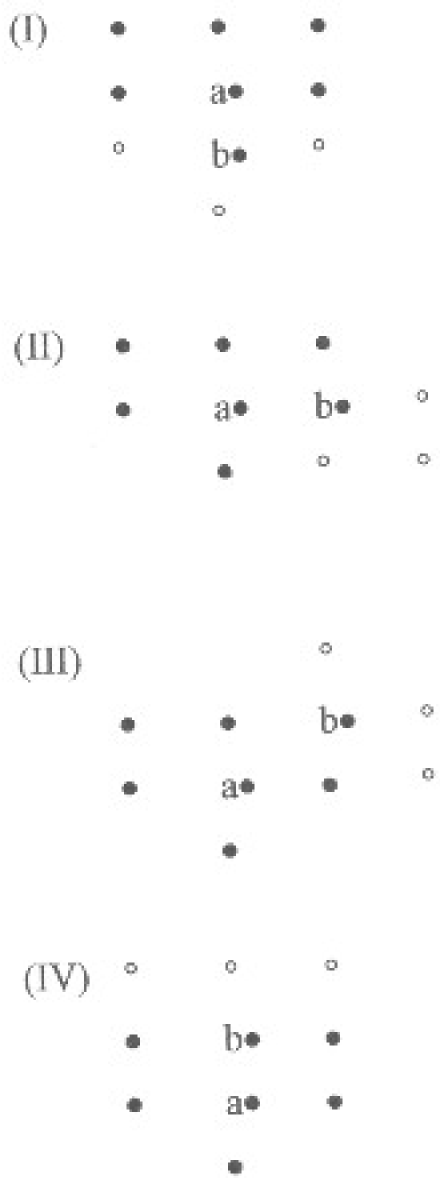

Fig. 4.7

Every set satisfies the theorem 3.10 and 3.11 and the expected negative result is obvious.□

Corollary 4.4 *There is not a topology in $\mathbf{Z}^2$ which is compatible with the v-connectivity induced by d_6.*

REFERENCES

1. J.M. Chassery; M.I. Chenin, Topologies on Discrete Spaces, J.C. Simon and R.M. Haralick (eds.), *Digital Image Processing*, D. Reidel Publishing Company, 59-66(1981)

2. J.M. Chassery, Connectivity and consecutivity in Digital Pictures, *Computer Graphics and Image Processing*, 9, 294-300(1979)

3. P.P. Das, P.P. Chakrabarti, B.N. Chatterji, Distance functions in digital geometry, *Inform.Sci.*, 42, 113-136(1987)

4. P.P. Das, B.N. Chatterji, Octagonal distances for digital pictures, *Inform.Sci.*, 50(1990)

5. P.P. Das, Best Simple Octagonal Distances in Digital Geometry, *Journal of Approximation Theory*, 68, 155-174(1992)

6. F. Hausdorff, *Teoria mnojestv*, Gostehizdat, Moskva, 1937

7. L.V. Kantorovici, G.P. Akilov, *Analyse Fonctionnelle*, MIR, Moscou, 1981

8. A. Rosenfeld, Arcs and curves in Digital Pictures, *Journal of the Association for Computing Machinery*, 20, 1, 81-87 (1973)

9. A. Rosenfeld, Three Dimensional Digital Topology, *Information and Control*, 50, 119-127(1981)

10. A. Rosenfeld, Digital Topology, *American Mathematical Monthly*,

621-630(1969)

11. A. Rosenfeld; J.L. Pfaltz, Distance Functions on Digital Pictures, *Pattern Recognition*, Pergamon Press, 1, 33-61(1968)

12. F. Wyse, et. al., Solution to Problem 5712, *American Mathematical Monthly*, 77, 1119(1970)

Received 11.07.1998

Gabriela Cristescu

“Aurel Vlaicu “ University of Arad

Bd.Revoluției 81,

2900-Arad

ROMANIA

E-mail:gcristescu@inext.ro