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On Dynamics of Solid-Fluid System

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Abstract

The existence and uniqueness of the solution for the problem of solid-fluid small perturbations from an uniform rotational motion around a horizontal fixed axis are proved. It is used the orthogonal projection method on closed subspaces, more precisely on the solenoidal functions space and the potential functions space, respectively.

1 Introduction

The following subspaces used in the hydrodynamics of the ideal, incompressible fluid are defined [1]:

$$\widetilde{G}^1(\Omega) = \left\{ \overrightarrow{v} \in \widetilde{L}_2(\Omega) \,|\, \overrightarrow{v} = \nabla\varphi,\ \varphi \in H^1(\Omega) \right\} \tag{1}$$

$$\widetilde{J}_0(\Omega) = \left\{ \overrightarrow{u} \in \widetilde{L}_2(\Omega) \,|\mathrm{div}\, \overrightarrow{u} = 0,\ u_n\, (= \overrightarrow{u}|_{\partial\Omega} \cdot \overrightarrow{n}) = 0 \quad on \quad \partial\Omega \right\} \tag{2}$$

where $H^1(\Omega)$ is the first order Sobolev space, div $\overrightarrow{u}$ is the generalized divergence and u_n is the generalized component defined by Green type formulas (γ - the trace operator for $f \in H^1(\Omega)$):

$$\int_\Omega \overrightarrow{v} \cdot \nabla\Phi \mathrm{d}\Omega + \int_\Omega \Phi \ \mathrm{div}\ \overrightarrow{v} \mathrm{d}\Omega = 0, \quad \forall \Phi \in C_0^\infty(\Omega)... \tag{3}$$

$$\int_\Omega \overrightarrow{u} \cdot \nabla f \mathrm{d}\Omega + \int_\Omega f \ \mathrm{div}\ \overrightarrow{u} \mathrm{d}\Omega = \int_{\partial\Omega} u_n \cdot \gamma f \mathrm{dS}, \quad \forall f \in H^1(\Omega) \tag{4}$$

$$\left(\overrightarrow{u} \in \widetilde{L}_2(\Omega), \quad \mathrm{div}\ \overrightarrow{u} \in L_2(\Omega),\ \gamma f \in H^{1/2}(\partial\Omega),\ u_n \in H^{-1/2}(\partial\Omega) \right)$$

Using these formulas, the decomposition of the space $\widetilde{L}_2(\Omega)$ of square integrable vector functions in an orthogonal sum, is proved:

$$\widetilde{L}_2(\Omega) = \widetilde{G}^1(\Omega) \oplus \widetilde{J}_0(\Omega) \tag{5}$$

The equations of the small motions of a solid-fluid system formed on a solid (in particular a plate) having a cavity Ω completely filled with an inviscid incompressible fluid are [3]:

$$\frac{\partial \overrightarrow{u}}{\partial t} + 2\overrightarrow{\omega}_0 \times \overrightarrow{u} + \frac{\mathrm{d}\overrightarrow{\omega}}{\mathrm{d}t} \times \overrightarrow{r} = -\frac{1}{\rho}\nabla p + \overrightarrow{f}(\overrightarrow{r}, t), \quad \mathrm{div}\, \overrightarrow{u} = 0 \quad in \quad \Omega \tag{6}$$

$$J\frac{\mathrm{d}\overrightarrow{\omega}}{\mathrm{d}t} + \overrightarrow{\omega}_0 \times (J\overrightarrow{\omega}) + \overrightarrow{\omega} \times (J\overrightarrow{\omega}_0) + \frac{\partial}{\partial t}\left(\rho \int_{\Omega} \overrightarrow{r} \times \overrightarrow{u}\,\mathrm{d}\Omega\right) + + \overrightarrow{\omega}_0 \times \left(\rho \int_{\Omega} \overrightarrow{r} \times \overrightarrow{u}\,\mathrm{d}\Omega\right) = \overrightarrow{M}_0(t) \tag{7}$$

with the initial and boundary conditions:

$u_n = 0 \;\; on \;\partial\Omega, \quad \overrightarrow{u}(\overrightarrow{r}, 0) = \overrightarrow{u}^{(0)}, \quad \overrightarrow{\omega}(0) = \overrightarrow{\omega}^{(0)}$

where $\overrightarrow{u}(\overrightarrow{r}, t)$ the relative velocity in the fluid and $\overrightarrow{\omega}(t)$ is the angular velocity of the solid-fluid system.

Considering a system of Cartesian coordinates fixed on the solid, namely $Oxyz$, J is the moment of inertia in $Oxyz$, $\overrightarrow{\omega}_0$ is the angular velocity of the system (around a fixed axis), p is the dynamic pressure ($p(\overrightarrow{r}, t) = P(\overrightarrow{r}, t) - p_0(\overrightarrow{r})$, P - the pressure in the fluid, p_0 - the pressure in the unperturbed state). It is admitted that $\overrightarrow{u}$, $\overrightarrow{\omega}$, p and the force $\overrightarrow{f}$ are first order small quantities (in the perturbations theory sense).

2 Small perturbations of the fluid in the case of a plate uniform rotation with the constant angular velocity around the horizontal axis Oz.

(the case when $\overrightarrow{\omega} = \overrightarrow{\omega}_0$ is given)

In the unperturbed state, the inviscid fluid is moving in the plate cavity Ω like a rigid solid. The velocity and pressure distribution are [2]:

$$\overrightarrow{v} = \overrightarrow{\omega}_0 \times \overrightarrow{r}, \quad p_0(x, y) = -\rho g x + \frac{1}{2}\rho\omega_0^2(x^2 + y^2) + p_a$$

$$(p_a = const. = p_0(0, 0))$$

The perturbed motion which little goes out from the uniform rotation (unperturbed above-mentioned) is represented through linear equations (the unknown functions are $\overrightarrow{u}$ and p; P (the fluid pressure) $= p_0 + p$):

$$\frac{\partial \overrightarrow{u}}{\partial t} + 2\omega_0(\overrightarrow{k} \times \overrightarrow{u}) = \overrightarrow{f} - \frac{1}{\rho}\nabla p, \quad \operatorname{div} \overrightarrow{u} = 0 \quad in \quad \Omega_T = (0,T) \times \Omega \tag{8}$$

$$\overrightarrow{u} \cdot \overrightarrow{n} = 0 \quad on \quad \partial\Omega \quad (or\ u_n = 0\ on\ \partial\Omega), \quad \overrightarrow{u}(\overrightarrow{r},0) = \overrightarrow{u}^{(0)}(\overrightarrow{r})$$

The orthogonal projection method on the subspace $\widetilde{J}_0(\Omega)$ (with the operator P_0) and on the subspace $\widetilde{G}^1(\Omega)$ (with the operator P_1) is applied and the following abstract operatorial equations are obtained:

$$\begin{cases} \frac{d\overrightarrow{u}}{dt} - A\overrightarrow{u} = P_0\overrightarrow{f}, \ with\ A\overrightarrow{u} = 2P_0\left[\omega_0(\overrightarrow{u} \times \overrightarrow{k})\right], \quad \overrightarrow{u}(\overrightarrow{r},0) = \overrightarrow{u}^{(0)} \\ -2\omega_0 P_1(\overrightarrow{u} \times \overrightarrow{k}) = P_1\overrightarrow{f} - \frac{1}{\rho}\nabla p \end{cases} \tag{9}$$

($P_1\overrightarrow{V} = 0$ if $\overrightarrow{V} \in \widetilde{J}_0(\Omega)$ and $P_1\overrightarrow{V} = \overrightarrow{V}$ if $\overrightarrow{V} \in \widetilde{G}^1(\Omega)$, $P_1(\overrightarrow{u} \times \overrightarrow{\omega_0}) = \nabla\varphi$, $\overrightarrow{u} \times \overrightarrow{\omega_0} = P_0(\overrightarrow{u} \times \overrightarrow{\omega_0}) + P_1(\overrightarrow{u} \times \overrightarrow{\omega_0}) = P_0(\overrightarrow{u} \times \overrightarrow{\omega_0}) + \nabla\varphi$ and $\int\limits_\Omega \overrightarrow{u} \cdot \nabla\varphi d\Omega = 0$). The potential component of the force $P_1\overrightarrow{f}$ has no influence on the velocity $\overrightarrow{u}$.

The properties of the Coriolis operator $A : \widetilde{J}_0(\Omega) \longrightarrow \widetilde{J}_0(\Omega)$ are proved. The operator A is antisymmetric and bounded.

The abstract solution for a first order evolution equation (Cauchy Problem) with the bounded operator A and $P_0\overrightarrow{f}$ - a continuous function is obtained using the theorem of the solution existence and uniqueness:

$$\overrightarrow{u}(t) = e^{tA}\overrightarrow{u}^{(0)} + \int\limits_0^t e^{(t-\tau)A}(P_0\overrightarrow{f})(\tau)d\tau; \quad \left(e^{tA} = \sum_{k=1}^{\infty}\frac{t^k}{k!}A^k\right) \tag{10}$$

Remark 1 *Considering that $\overrightarrow{f}(\overrightarrow{r},t) = 0$ and complexes Hilbert spaces, the solutions for a given problem are wanted in the natural oscillation form [ω, $\overrightarrow{u}(\overrightarrow{r})$, $p(\overrightarrow{r})$ - unknown]:*

$$\overrightarrow{u}(\overrightarrow{r},t) = e^{i\omega t}\overrightarrow{u}(\overrightarrow{r}), \quad p(\overrightarrow{r},t) = e^{i\omega t}p(\overrightarrow{r}) \tag{11}$$

With these solutions, the equation (8) in projection on the subspace $\widetilde{J}_0(\Omega)$, give the operatorial equation:

$$A\vec{u} = i\omega\vec{u}, \quad \vec{u} \in \tilde{J}_0(\Omega) \tag{12}$$

It follows (and it is demonstrated) that the spectrum of the operator A is pure imaginary and fills the interval [-2iω_0, 2iω_0]. The proper frecvency are real and $|\omega| \leq 2\omega_0$.

3 The perturbed motion of the gyrostat

(plate+fluid, $\vec{\omega}_0 = \omega_0\vec{k}$, $\vec{\omega} = \omega(t)\vec{k}$)(near the uniform rotation)

Let suppose that a little perturbation for the uniform rotation of the plate+fluid system (with the angular velocity $\vec{\omega}_0$) around the horizontal fixed axis, is made. Let $\vec{\omega} = \vec{\omega}_0 + \vec{\omega}_{pert}$ be the angular velocity of the gyrostat and $\vec{u}(\vec{r},t)$ - the relative velocity in the fluid from the cavity Ω. Supposing that $\vec{\omega}_{pert} \equiv \vec{\omega}(t)$ (having the fixed direction $Oz(\vec{k})$), $\vec{u}(\vec{r},t)$ and $\vec{f}(\vec{r},t)$ are small quantities (in the small perturbations theory sense), the equations of the unknown functions $\vec{u}$, $\vec{\omega}$, and p are:

$$\frac{\partial\vec{u}}{\partial t} + 2\vec{\omega}_0 \times \vec{u} + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} \times \vec{r} = -\frac{1}{\rho}\nabla p + \vec{f}(\vec{r},t), \quad \operatorname{div}\vec{u} = 0 \quad in \quad \Omega \tag{13}$$

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\partial}{\partial t}\left(\rho\int_{\Omega}\vec{r} \times \vec{u}\,\mathrm{d}\Omega\right)_{Oz} = M_{Oz}(t) \tag{14}$$

The scalar equations of the unsteady motion of the fluid are:

$$(P_1)\quad \begin{cases} \frac{\partial u}{\partial t} - 2\omega_0 v - y\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x \\ \frac{\partial v}{\partial t} + 2\omega_0 u + x\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + f_y \\ J\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\rho\int_{\Omega}(xv - yu)\mathrm{d}\Omega\right] = M_{Oz}(t) \end{cases} \tag{15}$$

with the initial and boundary conditions:

$$u_n = 0 \quad on \quad \partial\Omega \quad and \quad \vec{u}(\vec{r},0) = \vec{u}^{(0)}, \ \vec{\omega}(0) = \vec{\omega}^{(0)} \tag{16}$$

Supposing that the terms of the equation (13) belong to the space $\tilde{L}_2(\Omega) = \tilde{G}^1(\Omega) \oplus \tilde{J}_0(\Omega)$, the equation (13) is orthogonal projected on the subspace $\tilde{J}_0(\Omega)$ (of the solenoidal vector functions) and the following equation is obtained:

$$\frac{\partial}{\partial t}\left[\overrightarrow{u}+P_0(\overrightarrow{\omega}\times\overrightarrow{r})\right]+2i\omega_0 A\overrightarrow{u}=P_0\overrightarrow{f};\quad \left(A\overrightarrow{u}=iP_0(\overrightarrow{u}\times\overrightarrow{k})\right) \tag{17}$$

$$\frac{\partial}{\partial t}\left[J\omega+\rho\int\limits_{\Omega}(\overrightarrow{r}\times\overrightarrow{u})_{Oz}\mathrm{d}\Omega\right]=M_{Oz}(t) \tag{18}$$

The equation (18) results from (14); A is a symmetric and bounded operator because the bilinear form associated to the operator A is ($\forall\overrightarrow{u},\ \overrightarrow{v}\in\widetilde{L}_2\left(0,t_1;\widetilde{J}_0(\Omega)\right)$ with $\overrightarrow{u}\times\overrightarrow{e}_3=P_0(\overrightarrow{u}\times\overrightarrow{e}_3)+\nabla\psi,\ \psi\in\widetilde{G}^1(\Omega)$):

$$(A\overrightarrow{u},\overrightarrow{v})_{\widetilde{L}_2}=\int\limits_0^{t_1}\int\limits_{\Omega}\overrightarrow{v}\cdot\overline{A\overrightarrow{u}}\,\mathrm{d}\Omega\mathrm{d}t=(\overrightarrow{u},A\overrightarrow{v})\Longrightarrow A^*=A$$

$$\|A\overrightarrow{u}\|^2_{\widetilde{L}_2}=\int\limits_0^{t_1}\int\limits_{\Omega}\left|P_0(\overrightarrow{u}\times\overrightarrow{k})\right|^2\mathrm{d}\Omega\mathrm{d}t\le\int\limits_0^{t_1}\int\limits_{\Omega}|\overrightarrow{u}|^2\,\mathrm{d}\Omega\mathrm{d}t=\|\overrightarrow{u}\|^2_{\widetilde{L}_2}\Longrightarrow$$

$$\Longrightarrow\|A\|=\max\frac{\|A\overrightarrow{u}\|^2_{\widetilde{L}_2}}{\|\overrightarrow{u}\|^2_{\widetilde{L}_2}}\le 1\Longrightarrow\sigma(A)=[-1,1]$$

where the spectrum $\sigma(A)$ is continuous (fills the whole interval [-1,1]).

Let be the matrices:

$$v=\left\{\begin{array}{c}\overrightarrow{u}(\overrightarrow{r},t)\\ \\ \overrightarrow{\omega}(t)\end{array}\right\};\quad \widetilde{I}v=\left\{\begin{array}{c}\overrightarrow{u}+P_0(\overrightarrow{\omega}\times\overrightarrow{r})\\ \\ J\,\omega+\rho\int\limits_{\Omega}(\overrightarrow{r}\times\overrightarrow{u})_{Oz}\mathrm{d}\Omega\end{array}\right\}; \tag{19}$$

$$Bv=\left\{\begin{array}{c}2iA\overrightarrow{u}\\ \\ 0\end{array}\right\};\quad \varphi(t)=\left\{\begin{array}{c}P_0\overrightarrow{f}\\ \\ M_{Oz}(t)\end{array}\right\};\quad v^{(0)}=\left\{\begin{array}{c}\overrightarrow{u}^{(0)}\\ \\ \overrightarrow{\omega}^{(0)}\end{array}\right\}$$

where $\overrightarrow{u}(\overrightarrow{r},t)$ is a function which for every $t\in(0,t_1)$ has values in the space $\widetilde{J}_0(\Omega)$, $\overrightarrow{\omega}:(0,t_1)\longrightarrow\mathbb{R}^3$, $v:(0,t_1)\longrightarrow H=\widetilde{J}_0(\Omega)\times\mathbb{R}^3$. The space H is endowed with the norm:

$$\|v\|^2_H=\rho\int\limits_{\Omega}|\overrightarrow{u}|^2\,\mathrm{d}\Omega+|\overrightarrow{\omega}|^2,\quad\forall v\in H \tag{20}$$

(ρ is the density of the fluid).

Hence, the equations (17)-(18) are able to be written in the form of the following evolution equation:

$$\widetilde{I}\frac{\mathrm{d}v}{\mathrm{d}t} + \omega_0 Bv = \varphi(t), \quad v(0) = v^{(0)} \tag{21}$$

which has the unique solution:

$$v(t) = e^{t\widetilde{A}}v^{(0)} + \int_0^{t_1} e^{(t-\tau)\ \widetilde{A}}\widetilde{I}^{-1}\varphi(\tau)\mathrm{d}\tau, \quad \left(\widetilde{A} = -\omega_0\widetilde{I}^{-1}B\right) \tag{22}$$

The properties of the operators $\widetilde{I}$ and B are: the operator B are bounded (like the operator A) and the operator $\widetilde{I}$ is positive defined (which assure the existence and uniqueness of the inverse operator $\widetilde{I}^{-1}$) [3].

For proving the property of $\widetilde{I}$, the quadratic form was evaluated:

$$(\widetilde{I}v, v)_H = \rho\int_\Omega [\overrightarrow{u} + P_0(\overrightarrow{\omega} \times \overrightarrow{r})] \cdot \overrightarrow{u}\mathrm{d}\Omega + \left[J\omega + \rho\int_\Omega (\overrightarrow{r} \times \overrightarrow{u})_{Oz}\mathrm{d}\Omega\right] \cdot \omega =$$

$$= \rho\int_\Omega |\overrightarrow{u}|^2\,\mathrm{d}\Omega + \rho\int_\Omega \overrightarrow{u} \cdot [\overrightarrow{\omega} \times \overrightarrow{r} - \nabla\psi_1]\,\mathrm{d}\Omega + \rho\omega\int_\Omega (\overrightarrow{r} \times \overrightarrow{u})_{Oz}\mathrm{d}\Omega + J_{Oz}\omega^2 =$$

$$= \rho\int_\Omega |\overrightarrow{u}|^2\,\mathrm{d}\Omega + \rho\int_\Omega \overrightarrow{u} \cdot (\overrightarrow{\omega} \times \overrightarrow{r})\mathrm{d}\Omega + \rho\int_\Omega \overrightarrow{u} \cdot (\overrightarrow{\omega} \times \overrightarrow{r})\mathrm{d}\Omega + J_{Oz}\omega^2$$

$$J_{Oz}^{(f)}\overrightarrow{\omega} = \rho\int_\Omega \overrightarrow{r} \times (\overrightarrow{\omega} \times \overrightarrow{r})\mathrm{d}\Omega$$

$$J_{Oz}^{(f)}\omega^2 = \rho\int_\Omega [\overrightarrow{r} \times (\overrightarrow{\omega} \times \overrightarrow{r})] \cdot \overrightarrow{\omega}\mathrm{d}\Omega = \rho\int_\Omega (\overrightarrow{\omega} \times \overrightarrow{r}) \cdot (\overrightarrow{\omega} \times \overrightarrow{r})\mathrm{d}\Omega =$$

$$= \rho\int_\Omega |\overrightarrow{\omega} \times \overrightarrow{r}|^2\,\mathrm{d}\Omega$$

$$J_{Oz} = J_{Oz}^{(s)} + J_{Oz}^{(f)}$$

The following formulas is deduced:

$$(\widetilde{I}v,v)_H = J_{Oz}^{(s)}\omega^2 + \rho \int\limits_\Omega \left| \overrightarrow{u} + \omega(\overrightarrow{k} \times \overrightarrow{r}) \right|^2 d\Omega$$

The inertia matrix for the solid, $J_{Oz}^{(s)}$, is positive defined. Hence, it follows that the operator $\widetilde{I}$ is an operator strict positive:

$$(\widetilde{I}v,v)_H \geq 0, \quad (\widetilde{I}v,v)_H = 0 \Rightarrow \overrightarrow{\omega} = 0 \quad and \quad \overrightarrow{u} = 0$$

More precisely, the operator $\widetilde{I}$ is positive defined.

4 Final remarks

If the following conditions are fulfilled:

1) $\overrightarrow{u}^{(0)} \in \widetilde{J}_0(\Omega)$, $\quad \overrightarrow{\omega}^{(0)} \in \mathbb{R}^3$,

2) $M_{Oz}(t)$ is a continuous function from $\mathbb{R}^1$,

3) $\overrightarrow{f}(\overrightarrow{r},t)$ is a continuous function (which depends on t) with values in $\widetilde{L}_2(\Omega)$,

then the problem (13) - (14) has an unique continuous differentiable solution, $\{ \overrightarrow{u}(\overrightarrow{r},t); \overrightarrow{\omega}(t) \}$, which has the values from the space $H = \widetilde{J}_0(\Omega) \times \mathbb{R}^3$.

References

[1] BRADEAN R., FILIP Diana and BRADEANU P., *On dynamics of a plate having a cavity completely filled with an inviscid incompressible fluid*, submitted for publication in Révue Roumaine des Sciences Téchniques, Série de Mécanique Appliqué, Acad. Roum., Bucharest, (1997);

[2] BRADEANU P., *Metode variationale in matematica aplicata si hidrodinamica*, submitted for publication, (1997);

[3] KOPACEVSKI I. D., KREIN S. G., NGO ZUY CAN, *Operatorial Methods in Hydrodynamics. Evolution and Spectral Problems*, Nauka, Moscow, 1989, (in russian);

[4] LUKOVSKI I. A. et col., *Approximation Method for Fluid Dynamics in Bounded Cavities*, Kiev, Naukova Dumka, 1984, (in russian).

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