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*Dedicated to Professor Ion PĂVĂLOIU on his $60^{th}$ anniversary*

# A Galerkin Method for a Singularly Perturbed Bilocal Problem*

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**Abstract**

A bilocal singularly perturbed problem is solved using Galerkin's method in a space in which the test functions are weighted primitives of wavelets. This method provides a "good" numerical solution of this problem.

In the study of convection - diffusion problems, the following boundary value singularly perturbed problem appears:

$$(P)\begin{cases} -\varepsilon u''(x) + a(x)u'(x) = f(x), \text{ for } x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

where $0 < \varepsilon \ll 1$, $a(x) > \alpha > 0$ , $x \in [0,1]$ and functions $a$ and $f$ are sufficiently smooth.

The exact solution of problem $(P)$ has a boundary layer in $x = 1$. Because of its presence, certain numerical methods (finite element method, centered finite difference method) lead to numerical solutions with oscillations in the area of the boundary layer, abnormal from the physical point of view.

The piecewise polinomial test functions are replaced by wavelets, within the finite element method, in the work of Glowinski, Lawton, Ravachol and Tenenbaum [2]. Many examples provided show the great potential which wavelets have in the numerical solving of differential equations. Unfortunately, some disavantages may occur : the weak regularity of wavelets does not allow the use of small order wavelets; orthogonality of wavelets does not play a significant role.

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Disadvantages in the use of wavelets can be partially eliminated if primitives of wavelets as test functions ([4]) are used.

In the present paper, Galerkin's method is not applied, for problem $(P)$ (in the space $H_0^1[0,1]$). First, the space $H_0^1[0,1]$ turns "conveniently" into the space $GH_0^1[0,1]$, which is the image of $H_0^1[0,1]$ by $Gu := uog$, $u \in H_0^1[0,1]$ and $g : [0,1]- > [0,1]$ with $g(0) = 0$, $g(1) = 1$ and $\exists M > 0$, such that $0 \leq g'(y) \leq M$, $y \in [0,1]$. Problem $(P)$ is transcribed in $GH_0^1[0,1]$ and Galerkin's method is applied in order to solve the new problem. Weighted primitives of Haar's system are used as test functions (weighted primitives of Daubechies wavelets of the first order).

Accordingly, the problem in $GH_0^1[0,1]$ will have a Galerkin solution with attenuated oscillations in the area of the boundary layer. Getting back to problem $(P)$, it becomes out that a very good solution from the numerical point of view, is obtained. The numerical example fairly confirms it.

We consider the standard spaces. Let

$$L^2[0,1] := \{v : [0,1]- > R \ / \ v \text{ is measurable, and } \|v\|_{L^2[0,1]} < \infty\},$$

norm on $L^2[0,1]$ being:

$$\|v\|_{L^2[0,1]} := \left(\int_0^1 |v(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Let

$$H^1[0,1] := \{v \in L^2[0,1] \ / \ v^{(k)} \in L^2[0,1] \text{ for } k = 0,1\},$$

with norm

$$\|v\|_1 := \left(\int_0^1 |v(x)|^2\,dx + \int_0^1 |v'(x)|^2\,dx\right)^{\frac{1}{2}},$$

and seminorm

$$|v|_1 := \left(\int_0^1 |v'(x)|^2\,dx\right)^{\frac{1}{2}}$$

and subspace

$$H_0^1[0,1] := \{v \in H^1[0,1] \ / \ v(0) = v(1) = 0\}.$$

Seminorm $|.|_1$ is norm (equivalent to $\|.\|_1$ ) on the space $H_0^1[0,1]$.
Let $g : [0,1]- > [0,1]$ a differentiable function on $[0,1]$ so that:

$$\begin{aligned} g(0) &= 0, g(1) = 1, \\ \exists M &> 0 \; a.i. \; 0 \leq g'(y) \leq M, \text{ for } \forall y \in [0,1]. \end{aligned}$$

We note $J(y) = g'(y)$ and we define $GH_0^1[0,1]$ to be the image of space $H_0^1[0,1]$ by transformation $Gu := u \circ g$.

**Lemma 1**

$$\|v\|_{GH_0^1} \equiv \left\{ \int_0^1 \frac{1}{J(y)} |v'(y)|^2 dy \right\}^{\frac{1}{2}}$$

*is norm on the space $GH_0^1$ and the following inequality takes place :*
$\|v\|_{L^\infty[0,1]} \equiv \sup\{|v(y)| : 0 \leq y \leq 1\} \leq \|v\|_{GH_0^1}$ *for* $\forall v \in GH_0^1$.

Proof.

Because for $\forall v \in GH_0^1(0,1)$, $\exists u \in H_0^1(0,1)$ so that $v(y) = u(g(y))$ we have

$$\|v\|_{GH_0^1(0,1)} = \left( \int_0^1 \frac{1}{J(y)} |v'(y)|^2 dy \right)^{\frac{1}{2}} = \left( \int_0^1 J(y) |u'(g(y))|^2 dy \right)^{\frac{1}{2}} = |u|_1$$

so $\|.\|_{GH_0^1(0,1)}$ is norm on space $GH_0^1$.

Let $v \in GH_0^1(0,1) \Longrightarrow v$ is absolutely continuous, and $v(0) = 0$, thus,

$$v(y) = \int_0^y v'(t)dt = \int_0^y \sqrt{J(t)} \left( \frac{1}{\sqrt{J(t)}} v'(t) \right) dt, \text{ for } \forall y \in [0,1].$$

By use of Cauchy-Schwartz inequality, we obtain

$$|v(y)| \leq \left( \int_0^1 J(t)dt \right)^{\frac{1}{2}} \left( \int_0^1 \frac{1}{J(t)} |v'(t)|^2 dt \right)^{\frac{1}{2}} = \|v\|_{GH_0^1} \cdot \text{q.e.d.}$$

For the sake of simplicity, we consider $a(x) \equiv 1$ in $(P)$ and consequently we have the following singularly perturbed problem :

$$(P1)\begin{cases} -\varepsilon u''(x) + u'(x) = f(x), \text{ for } x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Making the change of variable $x = g(y)$ we obtain:

$$(P2)\begin{cases} -\varepsilon\left(\frac{1}{J(y)}v'(y)\right)' + v'(y) = J(y)F(y), \text{ for } x \in (0,1), \\ v(0) = v(1) = 0 \end{cases}$$

where $F(y) = f(g(y))$, and $v(y) = u(g(y))$.

Problem $(P1)$ has one solution,

$$(1)\begin{cases} u \in H_0^1(0,1), \text{such that} \\ \varepsilon\int_0^1 u'(x)w'(x)dx + \int_0^1 u'(x)w(x)dx = \int_0^1 f(x)w(x)dx, \ \forall w \in H_0^1(0,1). \end{cases}$$

and problem $(P2)$ will have one solution,

$$(2)\begin{cases} v \in GH_0^1(0,1), \text{such that} \\ \varepsilon\int_0^1 \frac{1}{J(y)}v'(y)w'(y)dy + \int_0^1 v'(y)w(y)dy = \int_0^1 F(y)J(y)w(y), \ \forall w \in GH_0^1(0,1). \end{cases}$$

(see [3]).

We will describe further the approximation scheme of the solution of problem (2).

Let $L > 0$ fixed natural number, and $\Pi^L : 0 = y_0 < y_1 < ... < y_N = 1$ a uniform division of interval $[0,1]$, where $N = 2^L$ and $y_j = \frac{j}{2^L}, 0 \le j \le 2^L$. We define set $V^L$ as being the subspace of $GH_0^1$ , which contains those functions $w$ satisfying,

$$(3)\ \left(\frac{1}{J(y)}w'(y)\right)' = 0 \text{ where } y_j < y < y_{j+1} \text{ for each } 0 \le j \le 2^L - 1.$$

In $V^L$ we consider the system of functions:

$$(4)\ \Phi_{L,k}(y) = \frac{\int_0^y J(s)\chi_{L,k-1}(s)ds}{\int_0^1 J(s)\chi_{L,k-1}(s)ds} - \frac{\int_0^y J(s)\chi_{L,k}(s)ds}{\int_0^1 J(s)\chi_{L,k}(s)ds}$$

for $1 \le k \le 2^L - 1$, where $\chi_{L,k}$ is characteristic function of interval $\left[\frac{k}{2^L}, \frac{k+1}{2^L}\right]$. Obviously $\Phi_{L,k} \in V^L$ for $1 \le k \le 2^L - 1$ and $\Phi_{L,k}(0) = \Phi_{L,k}(1) = 0$.

Because in addition,

$$\Phi_{L,k}(y_j) = \delta_{k,j},\ 1 \leq k \leq 2^L - 1,\ 0 \leq j \leq 2^L$$

every function $g \in V^L$ can be represented using base $\{\Phi_{L,k}\}_{k=1}^{2^L-1}$ thus

$$g(y) = \sum_{k=1}^{2^L-1} g\left(y_k\right) \Phi_{L,k}\left(y\right).$$

Consequently $\{\Phi_{L,k}\}, 1 \leq k \leq 2^L - 1$ form a base in $V^L$.

For $v \in GH_0^1(0,1)$, we define the interpolant $w_L^*$ from $V^L$ as being the unique element from $V^L$ which satisfies

$$w_L^*(y_j) = v(y_j),\ 0 \leq j \leq 2^L.$$

**Lemma 2** *Let $v$ be the solution of problem $(P2)$ and be $w_L^*$ the unique $V^L-$ interpolant . Then,*

$$\|w_L^* - v\|_{L^\infty[y_j,y_{j+1}]} \leq hg'(\xi_j)\left|\int_{x_j}^{x_{j+1}} u''(x)dx\right|,\ 0 \leq j \leq 2^L - 1.$$

*where $h = \frac{1}{2^L}, \xi_j \in (y_j, y_{j+1})$ and $x_j = g(y_j)$ for $0 \leq j \leq 2^L - 1$.*

Proof.

Consider a certain interval $[y_j, y_{j+1}]$, $0 \leq j \leq 2^L - 1$. Because $w_L^* - v$ cancels at the end of this interval, we obtain, integrating by parts :

$$\int_{y_j}^{y_{j+1}} \frac{1}{J(t)} \left\{w_L^{*\prime}(t) - v'(t)\right\}^2 dt = \int_{y_j}^{y_{j+1}} \frac{1}{J(t)} \left[w_L^{*\prime}(t) - v'(t)\right]\left[w_L^{*\prime}(t) - v'(t)\right] dt =$$

$$= \frac{1}{J(t)} \left[w_L^{*\prime}(t) - v'(t)\right]\left[w_L^*(t) - v(t)\right]\Big|_{y_j}^{y_{j+1}} - \int_{y_j}^{y_{j+1}} \left\{\frac{1}{J(t)}\left[w_L^{*\prime}(t) - v'(t)\right]\right\}' \left[w_L^*(t) - v(t)\right] dt =$$

because $w_L^*(y_j) - v(y_j) = 0$, $w_L^*(y_{j+1}) - v(y_{j+1}) = 0$ and $v, w_L^* \in GH_0^1(0,1) \Longrightarrow \exists u, u_L^* \in H_0^1(0,1)$ so that $v(y) = u(g(y))$, $w_L^*(y) = u_L^*(g(y)) \Longrightarrow v'(y) = J(y)u'(g(y))$, $w_L^{*\prime}(y) = J(y)u_L^{*\prime}(g(y)) \Longrightarrow \frac{1}{J(t)}\left[w_L^{*\prime}(y) - v'(y)\right] = J(y)\left[u_L^*(g(y)) - u'(g(y))\right]$

$$= -\int_{y_j}^{y_{j+1}} \left(\frac{1}{J(t)} w_L^{*\prime}(t)\right)' [w_L^*(t) - v(t)]\, dt + \int_{y_j}^{y_{j+1}} \left[\frac{1}{J(t)} v'(t)\right]' [w_L^*(t) - v(t)]\, dt =$$

$$(5) \quad \int_{y_j}^{y_{j+1}} \frac{v'(t) - J(t)}{\varepsilon} [w_L^*(t) - v(t)]\, dt \le \left| \int_{y_j}^{y_{j+1}} \frac{v'(t) - J(t)}{\epsilon} dt \right| \|w_L^* - v\|_{L^\infty[y_j, y_{j+1}]}.$$

For $\forall y \in [y_j, y_{j+1}], 0 \le j \le 2^L - 1$, we have :

$$w_L^*(y) - v(y) = \int_{y_j}^{y_{j+1}} [w_L^{*\prime}(t) - v'(t)]\, dt = \int_{y_j}^{y_{j+1}} \sqrt{J(t)} \frac{1}{\sqrt{J(t)}} [w_L^{*\prime}(t) - v'(t)]\, dt \le$$

$$\le \left(\int_{y_j}^{y_{j+1}} J(t) dt\right)^{\frac{1}{2}} \left\{ \int_{y_j}^{y_{j+1}} \frac{1}{J(t)} [w_L^{*\prime}(t) - v'(t)]^2 \right\}^{\frac{1}{2}}$$

$$\Longrightarrow \|w_L^* - v\|_{L^\infty[y_j, y_{j+1}]} \le \left\{ \int_{y_j}^{y_{j+1}} J(t) dt \right\}^{\frac{1}{2}} \left\{ \int_{y_j}^{y_{j+1}} \frac{1}{J(t)} [w_L^{*\prime}(t) - v'(t)]^2 dt \right\}^{\frac{1}{2}}.$$

Using this and inequality (5), we obtain:

$$\|w_L^* - v\|_{L^\infty[y_j, y_{j+1}]} \le \left(\int_{y_j}^{y_{j+1}} J(t) dt\right) \left| \int_{y_j}^{y_{j+1}} \frac{v'(t) - J(t)}{\varepsilon} dt \right| = hg'(\xi_j) \left| \int_{y_j}^{y_{j+1}} u''(x) dx \right|,$$

where $h = \frac{1}{2^L}$, $\xi_j \in (y_j, y_{j+1})$ and $x_j = g(y_j)$ for $0 \le j \le 2^L - 1$.

Let $w_L(g(y)) = w_L^*(y)$ for $\forall y \in [y_j, y_{j+1}]$; then, we have

$$u(x) - w_L(x) = u(g(y)) - w_L(g(y)) = v(y) - w_L^*(y)$$

$$\Longrightarrow |u(x) - w_L(x)| = |v(y) - w_L^*(y)| \le \|v - w_L^*\|_{L^\infty[y_j, y_{j+1}]}$$

$$\Longrightarrow \max_{x \in [x_j, x_{j+1}]} |u(x) - w_L(x)| \le \|v - w_L^*\|_{L^\infty[y_j, y_{j+1}]} \le hg'(\xi_j) \left| \int_{y_j}^{y_{j+1}} u''(x) dx \right|$$

and, therefore,

$$\|u - w_L\|_{L^\infty[x_j, x_{j+1}]} \le hg'(\xi_j)\left|\int_{y_j}^{y_{j+1}} u''(x)dx\right|, \text{ q.e.d.}$$

With a view to obtain an evaluation of approximation error by Galerkin method, we will compare Galerkin approximation to the approximation by $V^L - interpolant$ of solution of problem (P2).

**Theorem 3** *Let $v$ be the solution of problem $(P2)$ , let $w_L^{\#}$ be the Galerkin approximation of it, and $w_L^*$ its interpolant in space $V^L$. Then, the following inequalities*

$$\left\|w_L^{\#} - v\right\|_{L^\infty[0,1]} \le 2\left\|w_L^* - v\right\|_{L^\infty[0,1]},$$

*holds.*

Proof. We will use the fact that on every interval $(y_j, y_{j+1})$, $0 \le j \le 2^L - 1$, the functions from $V^L$ satisfy the differential equation

$$\left(\frac{1}{J(y)}w_L^{*\prime}(y)\right)' = 0.$$

Let $M_k = \int_0^1 \left\{\frac{1}{J(y)}w_L^{*\prime}(y)\Phi'_{L,k}(y) + w_L^{*\prime}(y)\Phi_{L,k}(y) - F(y)J(y)\Phi_{L,k}(y)\right\}dy$, $1 \le k \le 2^L - 1$. Because

$$\int_0^1 \left\{\frac{1}{J(y)}v'(y)\Phi'_{L,k}(y) + v'(y)\Phi_{L,k}(y) - F(y)J(y)\Phi_{L,k}(y)\right\}dy = 0,\ 1 \le k \le 2^L - 1,$$

it results that

$$(6)\quad M_k = \int_0^1 \left\{\frac{1}{J(y)}[w_L^{*\prime}(y) - v'(y)]\Phi'_{L,k}(y) + [w_L^{*\prime}(y) - v'(y)]\Phi_{L,k}(y)\right\}dy.$$

Similarly, we have

$$\int_0^1 \left\{\frac{1}{J(y)}w_L^{\#\prime}(y)\Phi'_{L,k}(y) + w_L^{\#\prime}(y)\Phi_{L,k}(y) - F(y)J(y)\Phi_{L,k}(y)\right\}dy = 0,$$

thus

$$(7)\quad M_k = \int_0^1 \left\{ \frac{1}{J(y)}[w_L^{*\prime}(y) - w_L^{\#\prime}(y)]\Phi'_{L,k}(y) + [w_L^{*\prime}(y) - w_L^{\#\prime}(y)]\Phi_{L,k}(y) \right\} dy.$$

We also have

$$\int_0^1 \frac{1}{J(y)}[w_L^{*\prime}(y) - v'(y)]\Phi'_{L,k}(y)dy =$$

$$= \int_0^1 \frac{1}{J(y)}[w_L^{*\prime}(y) - v'(y)]\left[\frac{J(y)\chi_{L,k}(y)}{\int_0^1 J(s)\chi_{L,k}(s)ds} - \frac{J(y)\chi_{L,k+1}(y)}{\int_0^1 J(s)\chi_{L,k+1}(s)ds}\right]dy =$$

$$= \frac{\int_{\frac{k-1}{2^L}}^{\frac{k}{2^L}}[w_L^{*\prime}(y) - v'(y)]dy}{\int_{\frac{k-1}{2^L}}^{\frac{k}{2^L}} J(s)ds} - \frac{\int_{\frac{k}{2^L}}^{\frac{k+1}{2^L}}[w_L^{*\prime}(y) - v'(y)]dy}{\int_{\frac{k}{2^L}}^{\frac{k+1}{2^L}} J(s)ds} =$$

$$= \frac{[w_L^*(y) - v(y)]\Big|_{\frac{k-1}{2^L}}^{\frac{k}{2^L}}}{\int_{\frac{k-1}{2^L}}^{\frac{k}{2^L}} J(s)ds} - \frac{[w_L^*(y) - v(y)]\Big|_{\frac{k}{2^L}}^{\frac{k+1}{2^L}}}{\int_{\frac{k}{2^L}}^{\frac{k+1}{2^L}} J(s)ds} = 0,$$

and using this in (6), we obtain

$$(8)\quad M_k = \int_0^1 [w_L^{*\prime}(y) - v'(y)]\Phi_{L,k}(y)dy.$$

Considering $w_L^* = \sum_{k=1}^{2^L-1} u_k^*\Phi_{L,k}$, and $w_L^\# = \sum_{k=1}^{2^L-1} u_k^\#\Phi_{L,k}$, from (8) we obtain :

$$\sum_{k=1}^{2^L-1}(u_k^* - u_k^\#)M_k = \sum_{k=1}^{2^L-1}(u_k^* - u_k^\#)\int_0^1 [w_L^{*\prime}(y) - v'(y)]\Phi_{L,k}(y)dy =$$

$$= \int_0^1 \sum_{k=1}^{2^L-1}(u_k^* - u_k^\#)[w_L^{*\prime}(y) - v'(y)]\Phi_{L,k}(y)dy = \int_0^1 [w_L^*(y) - w_L^\#(y)][w_L^{*\prime}(y) - v'(y)]dy$$

and similarly, using (7)

$$(9)\quad \sum_{k=1}^{2^L-1} (u_k^* - u_k^{\#}) M_k =$$

we find

$$= \sum_{k=1}^{2^L-1} (u_k^* - u_k^{\#}) \int_0^1 \left\{ \frac{1}{J(y)} [w_L^{*\prime}(y) - w_L^{\#\prime}(y)] \Phi'_{L,k}(y) + [w_L^{*\prime}(y) - v'(y)] \Phi_{L,k}(y) \right\} dy =$$

$$= \int_0^1 \left\{ \frac{1}{J(y)} [w_L^{*\prime}(y) - w_L^{\#\prime}(y)]^2 + [w_L^{*\prime}(y) - v'(y)][w_L^*(y) - w_L^{\#}(y)] \right\} dy.$$

From (8) and (9) it results :

$$\int_0^1 [w_L^*(y) - w_L^{\#}(y)][w_L^{*\prime}(y) - v'(y)] dy =$$

$$= \int_0^1 \frac{1}{J(y)} [w_L^{*\prime}(y) - w_L^{\#\prime}(y)]^2 dy + \int_0^1 [w_L^{*\prime}(y) - w_L^{\#\prime}(y)][w_L^*(y) - w_L^{\#}(y)] dy$$

but

$$\int_0^1 [w_L^*(y) - w_L^{\#}(y)][w_L^{*\prime}(y) - w_L^{\#\prime}(y)] dy = \frac{1}{2}\, w_L^*(y) - w_L^{\#}(y) \Big|_0^1 = 0.$$

$$\Longrightarrow \int_0^1 [w_L^*(y) - w_L^{\#}(y)][w_L^{*\prime}(y) - v'(y)] dy = \int_0^1 \frac{1}{J(y)} [w_L^{*\prime}(y) - w_L^{\#\prime}(y)]^2 dy$$

$$\Longrightarrow \left\| w_L^* - w_L^{\#} \right\|_{GH_0^1}^2 = \int_0^1 [w_L^*(y) - w_L^{\#}(y)][w_L^{*\prime}(y) - v'(y)] dy =$$

$$= [w_L^*(y) - w_L^{\#}(y)][w_L^*(y) - v(y)] \Big|_0^1 - \int_0^1 [w_L^{*\prime}(y) - w_L^{\#\prime}(y)][w_L^*(y) - v(y)] dy =$$

$$= \int_0^1 [w_L^{\#\prime}(y) - w_L^{*\prime}(y)][w_L^*(y) - v(y)] dy \le \left( \int_0^1 [w_L^{\#\prime}(y) - w_L^{*\prime}(y)]^2 \right)^{\frac{1}{2}} \left( \int_0^1 [w_L^*(y) - v(y)]^2 dy \right)^{\frac{1}{2}} =$$

$$= \left( \int_0^1 J(y) \frac{1}{J(y)} [w_L^{\#\prime}(y) - w_L^{*\prime}(y)]^2 \right)^{\frac{1}{2}} \|w_L^* - v_L\|_{L^2[0,1]} \leq$$

$$\leq \left( \int_0^1 J(y) dy \right)^{\frac{1}{2}} \left( \int_0^1 \frac{1}{J(y)} [w_L^{\#\prime}(y) - w_L^{*\prime}(y)]^2 \right)^{\frac{1}{2}} \|w_L^* - v_L\|_{L^2[0,1]} =$$

$$= \left\| w_L^{\#} - w_L^* \right\|_{GH_0^1} \|w_L^* - v_L\|_{L^2[0,1]}$$

$$\Longrightarrow \left\| w_L^{\#} - w_L^* \right\|_{GH_0^1} \leq \|w_L^* - v_L\|_{L^2[0,1]}.$$

We have

$$\|w_L^* - v_L\|_{L^2[0,1]} \leq \|w_L^* - v_L\|_{L^\infty[0,1]}$$

$$\Longrightarrow \left\| w_L^{\#} - v \right\|_{L^\infty[0,1]} \leq \left\| w_L^{\#} - w_L^* \right\|_{L^\infty[0,1]} + \|w_L^* - v\|_{L^\infty[0,1]} \leq \left\| w_L^{\#} - w_L^* \right\|_{GH_0^1} + \|w_L^* - v\|_{L^\infty[0,1]} \leq$$

$$\leq \|w_L^* - v_L\|_{L^2[0,1]} + \|w_L^* - v\|_{L^\infty[0,1]} \leq 2 \|w_L^* - v\|_{L^\infty[0,1]}\ .q.e.d.$$

We similarly define space $V_{L-1}$, and within this space $\{\Phi_{L-1,k}\}_{k=1}^{2^{L-1}-1}$ is base, where

$$\Phi_{L-1,k}(y) = \frac{\int_0^y J(s)\chi_{L-1,k-1}(s)ds}{\int_0^1 J(s)\chi_{L-1,k-1}(s)ds} - \frac{\int_0^y J(s)\chi_{L-1,k}(s)ds}{\int_0^1 J(s)\chi_{L-1,k}(s)ds}.$$

We have :

$$\Phi_{L-1,k}(y) = H_0^{L-1,k}\Phi_{L,2k-1}(y) + H_1^{L-1,k}\Phi_{L,2k}(y) + H_2^{L-1,k}\Phi_{L,2k+1}(y),$$

where

$$H_0^{L-1,k} = \frac{\int_{\frac{k-1}{2^{L-1}}}^{\frac{k-\frac{1}{2}}{2^{L-1}}} J(s)ds}{\int_{\frac{k-1}{2^{L-1}}}^{\frac{k}{2^{L-1}}} J(s)ds}; H_1^{L-1,k} = 1; H_2^{L-1,k} = \frac{\int_{\frac{k+\frac{1}{2}}{2^{L-1}}}^{\frac{k+1}{2^{L-1}}} J(s)ds}{\int_{\frac{k}{2^{L-1}}}^{\frac{k+1}{2^{L-1}}} J(s)ds}.$$

Obviously $V^{L-1} \subset V^L$.
Let $\Psi_{L-1,k}(y) = \Phi_{L,2k-1}(y)$, $k = 1, ..., 2^{L-1}$ and

$$W^{L-1} = span\{\Psi_{L-1,k} / \ k = 1, ..., 2^{L-1}\}.$$

Functions $\Psi_{L-1,k}(y)$ have as support the interval $\left[\frac{2k-2}{2^L}, \frac{2k}{2^L}\right]$ and obviously $\mathrm{supp}\Psi_{L-1,k} \bigcap \mathrm{supp}\Psi_{L-1,l} = \Phi$ if $k \neq l$. System $\{\Psi_{L-1,k}\}_{k=1}^{2^{L-1}}$ forms a base of space $W^{L-1}$ and we also have $W^{L-1} \subset V^{L-1}$.

We will show that $V^{L-1} \bigoplus W^{L-1} = V^L$. The fact that $V^{L-1} + W^{L-1} = V^L$ is obvious; we will further prove that

$$V^{L-1} \perp W^{L-1}.$$

We have

$$\int_0^1 \frac{1}{J(y)} \Psi'_{L-1,k}(y) \Phi'_{L-1,k}(y) dy = \int_{\frac{k-1}{2^{L-1}}}^{\frac{k-\frac{1}{2}}{2^{L-1}}} \frac{J(y)}{\int_{\frac{k-1}{2^{L-1}}}^{\frac{k-\frac{1}{2}}{2^{L-1}}} J(s)ds \int_{\frac{k-1}{2^{L-1}}}^{\frac{k}{2^{L-1}}} J(s)ds} dy -$$

$$- \int_{\frac{k-\frac{1}{2}}{2^{L-1}}}^{\frac{k}{2^{L-1}}} \frac{J(y)}{\int_{\frac{k-\frac{1}{2}}{2^{L-1}}}^{\frac{k}{2^{L-1}}} J(s)ds \int_{\frac{k-1}{2^{L-1}}}^{\frac{k}{2^{L-1}}} J(s)ds} dy =$$

$$= \frac{1}{\int_{\frac{k-1}{2^{L-1}}}^{\frac{k}{2^{L-1}}} J(s)ds} - \frac{1}{\int_{\frac{k-1}{2^{L-1}}}^{\frac{k}{2^{L-1}}} J(s)ds} = 0.$$

Similarly,

$$\int_0^1 \frac{1}{J(y)} \Psi'_{L-1,k+1}(y) \Phi'_{L-1,k}(y) dy = 0.$$

Consequently, subspaces $V^{L-1}$ and $W^{L-1}$ are orthogonal related to the scalar product,

$$(f, g)_{GH_0^1} := \int_0^1 \frac{1}{J(y)} f'(y) g'(y) dy.$$

which means that the sum is direct..

The procedure can continue, leading finally to decomposition of space $V^L$, so that :

$$V^L = V^1 \bigoplus W^1 \bigoplus W^2 \bigoplus ... \bigoplus W^{L-1}.$$

This decomposition allows the consideration of another base in space $V^L$, namely :

$$\{\Psi_{jk}\}_{\substack{k=1,2^j-1 \\ j=0,L-1}}$$

where $\Psi_{jk}(y) = \Phi_{j+1,2k-1}(y)$.

We also have

$$\Psi_{jk}(y) = \begin{cases} \dfrac{\int_{\frac{k-1}{2^j}}^{y} J(s)\psi_{jk}(s)ds}{\int_{\frac{k-1}{2^j}}^{\frac{k-\frac{1}{2}}{2^j}} J(s)\psi_{jk}(s)ds}, \text{ if } \ y \in \left[\frac{k-1}{2^j}, \frac{k-\frac{1}{2}}{2^j}\right] \\ \dfrac{\int_{y}^{\frac{k}{2^j}} J(s)\psi_{jk}(s)ds}{\int_{\frac{k-\frac{1}{2}}{2^j}}^{\frac{k}{2^j}} J(s)\psi_{jk}(s)ds}, \text{ if } \ y \in \left[\frac{k-\frac{1}{2}}{2^j}, \frac{k}{2^j}\right] \\ 0, \ in\ rest \end{cases}$$

where $\psi_{jk}(y) = \sqrt{2^j}\psi(2^j y - k)$ and $\psi$ is Haar's function

$$\psi(x) = \begin{cases} 1 \ if\ 0 \le x < \frac{1}{2} \\ -1 \ if\ \frac{1}{2} \le x < 1 \\ 0 \ otherwise. \end{cases}.$$

**Numerical example.**

We consider the following problem :

$$\begin{aligned} -\varepsilon u''(x) + u'(x) &= 1, \text{ for } x \in (0,1) \\ u(0) &= u(1) = 0. \end{aligned}$$

which have the solution $u(x) = \frac{\exp(\frac{x}{\varepsilon}) - \exp(\frac{1}{\varepsilon})}{1 - \exp(\frac{1}{\varepsilon})} + x - 1$.

We do the change of variable $x = g(y)$ where $g(y) = 1 - (1-y)^{p+1}$.

In the Figures 1., 2., 3. the exact solution and the approximate solution are presented for $p = 0, p = 2, p = 4$ and $N = 4, \varepsilon = 0.0001$.

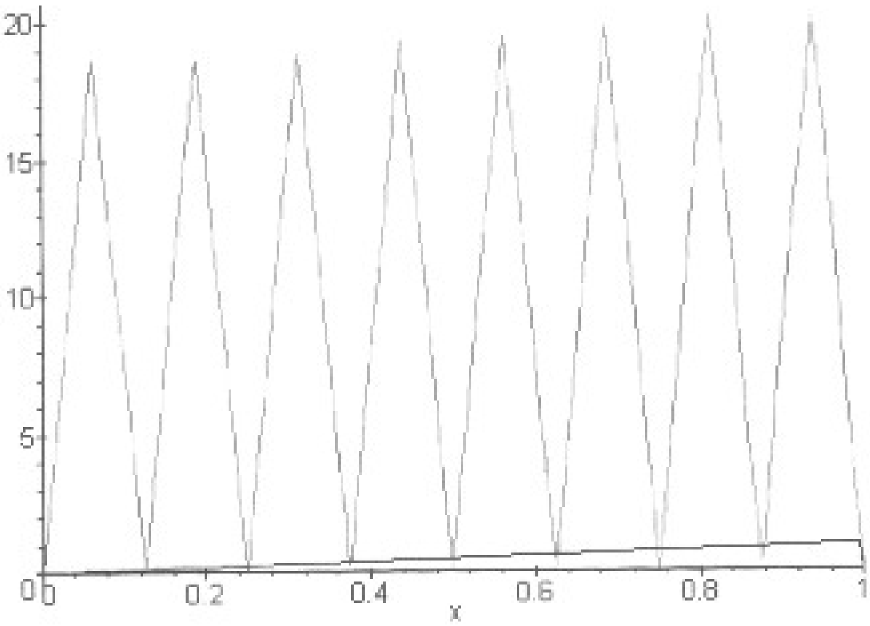


Figure 1:

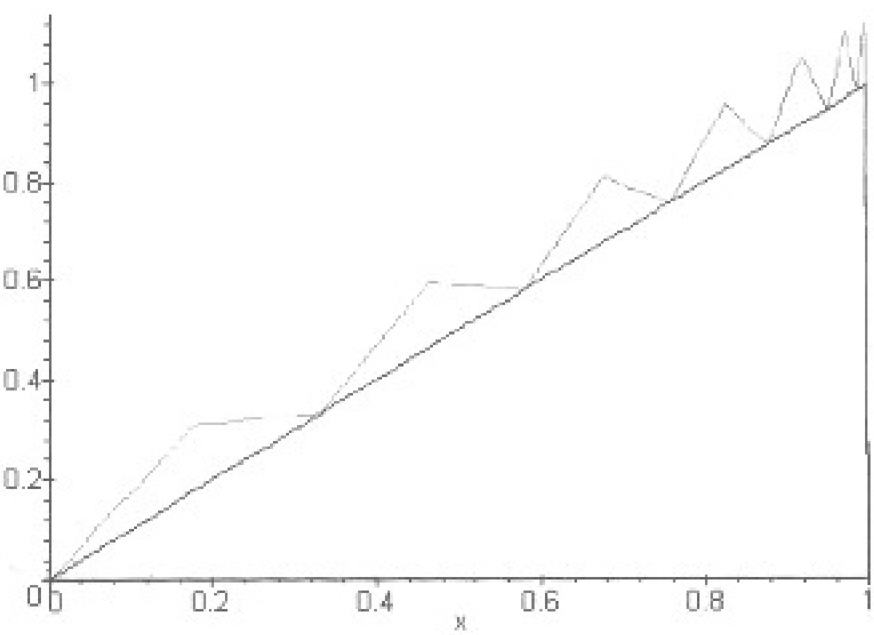


Figure 2:

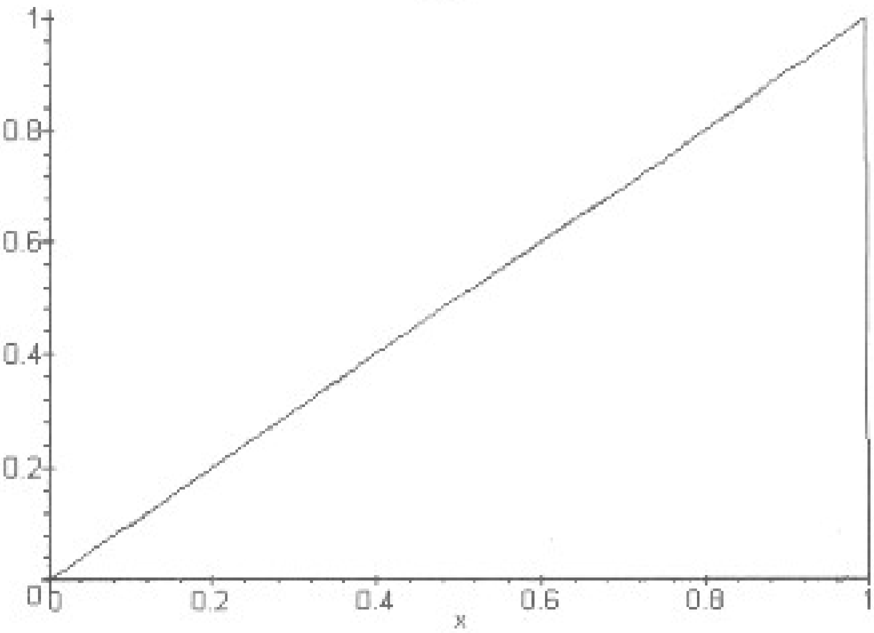


Figure 3:

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