Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XV(1999), Nr. 1-2, 103-110

*Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary*

# Monotone sequences for approximating the solutions of equations

Ion Păvăloiu

## 1 Introduction.

We shall consider in the following the Aitken–Steffensen–like methods and some conditions under which they generate bilateral sequences for the approximation of the solutions of the scalar equations.

Let $I = [a, b] \subset \mathbb{R}$, $a < b$, be an interval of the real axis and consider the equation

$$f(x) = 0, \tag{1.1}$$

where $f : I \to \mathbb{R}$. Let moreover,

$$\begin{aligned} x - g_1(x) &= 0 \\ x - g_2(x) &= 0, \end{aligned} \tag{1.2}$$

with $g_2, g_1 : I \to \mathbb{R}$ be other two equations.

*We shall assume that if $\overline{x}$ is a root of* (1.1)*, then it also satisfies both equations from* (1.2).

The Aitken–Steffensen method consists in the construction of the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$, $(g_2(x_n))_{n\geq 0}$ generated by the following iterative process:

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(g_1(x_n)); f]}, \quad n = 0, 1, \ldots, \quad x_0 \in I, \tag{1.3}$$

where $[u, v; f]$ denotes the first order divided difference of $f$ on the points $u$ and $v$.

The second order divided differences of $f$ will be denoted by $[u, v, w; f]$.

In this paper we shall show that in the study of the convergence of the sequences generated by (1.3), an important role is played by the hypothesis of convexity on the function $f$. We bring some completions and specifications to the results obtained in [5]–[7].

Concerning the convexity and the monotonicity of the functions we shall consider the following definitions (see, for example, [3, p.288–299 and p.327]).

**Definition 1.1** *The function $g : I \to \mathbb{R}$ is called increasing (nondecreasing, decreasing, resp. nonincreasing) on the interval $I$ if for all $x, y \in I$, it follows that $[x, y; g] > 0$ $(\geq 0, < 0,$ resp. $\leq 0)$.*

**Definition 1.2** *The function $g : I \to \mathbb{R}$ is called convex (nonconcave, concave, resp. nonconvex) if for all $x, y, z \in I$ it follows that $[x, y, z; g] > 0$ $(\geq 0, < 0,$ resp. $\leq 0)$.*

Some of the usual properties of the convex functions will be used in the following, and we remind them without proof (see, e.g. [3, pp.288–299]).

Denote $sg_{x_0}(x) = [x_0, x; g]$, $x \in I \setminus \{x_0\}$, the slope of the function $g$ at $x_0$. The following results hold:

**Proposition 1.1** *Let $g : I \to \mathbb{R}$ be an arbitrary function and $x_0 \in I$.*

*1. If $g$ is convex on $I$ then $sg_{x_0}$ is increasing on $I \setminus \{x_0\}$.*

*2. If $g$ is nonconcave on $I$, then $sg_{x_0}$ is nondecreasing on $I \setminus \{x_0\}$.*

**Proposition 1.2** *If $g : ]a, b[ \to \mathbb{R}$ is nonconcave, then $g$ admits the left derivative $g'_l(x)$ and the right derivative $g'_r(x)$ at any point $x \in ]a, b[$. Moreover, the functions $g'_l(x)$ and $g'_r(x)$ are nondecreasing on $]a, b[$ and $g'_l(x) \leq g'_r(x)$ for all $x \in ]a, b[$.*

**Proposition 1.3** *If $g : I \to \mathbb{R}$ is a convex function on $I$ then*

*1. the function $g$ is continuous at any point $x \in int(I)$;*

*2. the function $g$ satisfies the Lipschitz condition on any compact interval contained by $I$;*

*3. the function $g$ is derivable on $I$ excepting a subset of $I$ at most countable.*

**Proposition 1.4** *Let $g : int(I) \to \mathbb{R}$. The following statements are equivalent:*

*1. the function $g$ is convex on $int(I)$;*

*2. for any $x \in int(I)$ there exists the left derivative of $g$ at $x$, $g'_l(x)$, which is finite and is increasing as a function on $int(I)$;*

*3. for any $x \in int(I)$, there exists the right derivative of $g$ at $x$, $g'_r(x)$, which is finite and is increasing as a function on $int(I)$.*

Taking into account the properties expressed in propositions 1.1–1.4, we are interested in the present note to simplify the hypotheses requested in [5]–[7]. As we shall see, the convexity properties of the function $f$ from equation (1.1) play an essential role in the construction of the functions $g_1$ and $g_2$ from (1.2).

# 2 The monotonicity of the sequences generated by the Aitken-Steffensen method.

We shall consider the following hypotheses concerning the functions $f, g_1$ and $g_2$ :

*(a) the function $f$ is convex on $I$;*

*(b) the functions $g_1$ and $g_2$ are continuous on $I$;*

*(c) the function $g_1$ is increasing on $I$;*

*(d) the function $g_2$ is decreasing on $I$;*

*(e) equation (1.1) has a unique solution $\bar{x} \in I$;*

*(f) for any $x, y \in I$ it follows that $0 < [x, y; g_1] \leq 1$.*

Concerning the convergence of the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ and $(g_2(g_1(x_n)))_{n\geq 0}$, the following result holds.

**Theorem 2.1** *If the functions $f, g_1, g_2$ satisfy conditions $(a)-(f)$ and, moreover,*

*$i_1$. the function $f$ is increasing on $I$;*

*$ii_1$. there exists $x_0 \in I$ such that $f(x_0) < 0$ and $g_2(g_1(x_0)) \in I$,*

*then the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$, $(g_2(g_1(x_n)))_{n\geq 0}$ generated by (1.3), with the initial approximation $x_0$ considered above, have the following properties:*

*$j_1$. the sequences $(x_n)$ and $(g_1(x_n))$ are increasing and bounded;*

*$jj_1$. the sequence $(g_2(g_1(x_n)))_{n\geq 0}$ is decreasing and bounded;*

*$jjj_1$. $\lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = \bar{x}$*

*$jv_1$. the following relations hold:*

$$x_n \leq g_1(x_n) \leq \bar{x} \leq g_2(g_1(x_n)), \quad n = 0, 1, \ldots$$
$$\max\{\bar{x} - x_{n+1}, g_2(g_1(x_n)) - \bar{x}\} \leq g_2(g_1(x_n)) - x_{n+1}, \quad n = 0, 1, \ldots$$

**Proof.** Since $f$ is increasing on $I$, $f(x_0) < 0$, and $\bar{x}$ is the unique solution of $f(x) = 0$ on $I$, it follows that $x_0 < \bar{x}$. By c) and f), for $x < y$ we get $g_1(y) - g_1(x) \le y - x$. Now, for $y = \bar{x}$ one obtains $x - g_1(x) \le 0$ when $x < \bar{x}$ and $x - g_1(x) \ge 0$ when $x > \bar{x}$. By c) and $x_0 < \bar{x}$ it follows $g_1(x_0) < g_1(\bar{x})$, i.e. $g_1(x_0) < \bar{x}$. Since $x_0 < \bar{x}$, one gets $x_0 \le g_1(x_0)$. By d) and $g_1(x_0) < \bar{x}$ it results $g_2(g_1(x_0)) > g_2(\bar{x})$, i.e. $g_2(g_1(x_0)) > \bar{x}$. By $i_1$) and $g_1(x_0) < \bar{x}$ it results $f(g_1(x)) < 0$. Hypothesis $i_1$) also implies $[g_1(x_0), g_2(g_1(x_0)); f] > 0$, whence, by (1.3), one obtains $x_1 > g_1(x_0)$.

It can be easily verified that the following identities hold for all $x, y, z \in I$:

$$(2.1)\quad g_1(x) - \frac{f(g_1(x))}{[g_1(x), g_2(g_1(x)); f]} = g_2(g_1(x)) - \frac{f(g_2(g_1(x)))}{[g_1(x), g_2(g_1(x)); f]}$$

$$(2.2)\qquad f(z) = f(x) + [x, y; f](z - x) + [x, y, z; f](z - x)(z - y).$$

Since $g_2(g_1(x_0)) > \bar{x}$, it follows $f(g_2(g_1(x_0))) > 0$ and using (2.1) one obtains $x_1 < g_2(g_1(x_0))$. Now, if in (2.2) we set $z = x_1$, $x = g_1(x_0)$, $y = g_2(g_1(x_0))$ and we take into account (1.3) we get

$$f(x_1) = [g_1(x_0), g_2(g_1(x_0)), x_1; f](x_1 - g_1(x_0))(x_1 - g_2(g_1(x_0))).$$

But $f$ is a convex function, so $f(x_1) < 0$ and consequently $x_1 < \bar{x}$.

Summarizing, we have obtained the following relations

$$x_0 \le g_1(x_0) \le x_1 < \bar{x} < g_2(g_1(x_0)).$$

It remains to prove that $x_1$ satisfies hypothesis $iii_1$., and the above reasoning may be repeated.

Since $g_2$ is decreasing, $g_1$ is increasing and $x_0 < x_1$, the following inequalities are true: $g_1(x_0) < g_1(x_1)$, $g_2(g_1(x_0)) > g_2(g_1(x_1))$.

From $x_1 < \bar{x} \Rightarrow g_2(g_1(x_1)) > g_2(g_1(\bar{x}))$, i.e. $g_2(g_1(x_1)) > \bar{x}$, which shows that $g_2(g_1(x_1)) \in I$.

Consider now $x_n \in I$ with $f(x_n) < 0$ and $g_2(g_1(x_n)) \in I$. If in the above reasoning we take $x_0 = x_n$ we obtain

$$(2.3)\qquad x_n \le g_1(x_n) < x_{n+1} < \bar{x} < g_2(g_1(x_n)), \quad n = 0, 1, \ldots,$$

and so the affirmations $j_1$, $jj_1$ and $jv_1$ of the theorem are proved. In order to prove $jjj_1$ we denote $l_1 = \lim x_n$, $l_2 = \lim g_1(x_n)$ and $l_3 = \lim g_2(g_1(x_n))$ and we shall prove that $l_1 = l_2 = l_3 = \bar{x}$. Indeed, by (2.3) and (b) we get

$$l_1 \le g_1(l_1) \le l_1 \le \bar{x} \le g_2(g_1(l_1)),$$

i.e. $g_1(l_1) = l_1$ and so $l_1 \leq \bar{x} \leq g_2(l_1)$. Since $f$ is convex on $I$, Proposition 1.3 assures that $f$ is continuous in $l_1$, and by (1.3), passing to limit it follows $f(l_1) = 0$, i.e. $l_1 = \bar{x}$.

The inequality $g_1(l_1) = \bar{x}$ implies $l_2 = \bar{x}$.

Finally, $l_3 = g_2(l_1) \geq \bar{x} \Rightarrow f(g_2(l_1)) \geq 0$, and since $l_1 \leq g_2(l_1)$ and, at the same time, (2.1) implies $l_1 \geq g_2(l_1)$, we obtain $l_1 = g_2(l_1) = l_3$. □

Analogous results hold in the case when $f$ is decreasing and convex, or increasing, resp. decreasing and concave (see [7]).

# 3 The Steffensen method.

This method is obtained from (1.3) for $g_1(x) = x$ for all $x \in I$. For the sake of simplicity we shall denote in this section $g_2 = g$. So, the Steffensen method reads as

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]}, \quad n = 0, 1, \ldots, \quad x_0 \in I. \tag{3.1}$$

We observe that the hypotheses (b), (c) and (f) from the previous section are automatically satisfied for the function $g_1$ we have considered here.

Concerning the functions $f$ and $g$ it remains here to make the following assumptions:

$(a_1)$ *the function $f$ is convex on $I$;*

$(b_1)$ *the function $g$ is decreasing and continuous on $I$;*

$(c_1)$ *equations (1.1) and $x - g(x) = 0$ have each a unique solution $\bar{x} \in \operatorname{int} I$, which is the same.*

We obtain the following consequences concerning the converge of the method (3.1):

**Corollary 3.1** *If the functions $f$ and $g$ obey $(a_1)$–$(c_1)$ and, moreover, $f$ is increasing on $I$, there exists $f'(\bar{x})$ and the point $x_0$ in (3.1) may be chosen such that $f(x_0) < 0$ and $g(x_0) \in I$, then the sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ verify the following properties:*

$j_2$. *the sequence $(x_n)_{n\geq 0}$ is increasing and bounded;*

$jj_2$. *the sequence $(g(x_n))_{n\geq 0}$ is decreasing and bounded;*

$jjj_2$. $\lim x_n = \lim g(x_n) = \bar{x}$;

$jv_2$. $x_n \leq \bar{x} \leq g(x_n), \quad n = 0, 1, \ldots;$

$v_2$. $\max\{\bar{x} - x_n, g(x_n) - \bar{x}\} \leq g(x_n) - x_n, \quad n = 0, 1, \ldots$

We shall assume in the following that the function $f$ from equation (1.1) has the form $f(x) = x - g(x)$. In this case (3.1) becomes

$$(3.2) \quad x_{n+1} = x_n - \frac{(x_n - g(x_n))^2}{g(g(x_n)) - 2g(x_n) + x_n}, \quad n = 0, 1, \ldots, \quad x_0 \in I.$$

Concerning the convergence of these iterates we obtain from Corollary 3.1 the following result

**Corollary 3.2** *If $g$ is increasing and concave on $I$, equation $x - g(x) = 0$ has a unique solution $\bar{x} \in \operatorname{int}(I)$, there exists $g'(\bar{x})$ and the initial approximation is chosen such that $x_0 < g(x_0)$, with $g(x_0) \in I$, then the sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ generated by (3.2) verify the conclusions of Corollary 3.1.*

**Proof.** Since $g$ is decreasing on $I$, it follows that for any $x, y \in I$ we have $[x, y; g] < 0$ and so $1 - [x, y; g] > 0$, i.e. $[x, y; f] > 0$ for all $x, y \in I$, which implies that $f$ is increasing. On the other hand, for all $x, y, z \in I$ we have that $[x, y, z; f] = -[x, y, z; g]$, and since $g$ is concave we obtain that $f$ is convex. One can see that the hypotheses of Corollary 3.1 are satisfied. □

# 4 Applications.

In this section we shall show that the functions $g_1, g_2$ (resp. $g$) from the auxiliary equations (1.2) (resp. $x - g(x) = 0$) may be determined in different ways, under convexity and monotonicity assumptions on the function $f$ from (1.1), such that the essential hypotheses of Theorem 2.1, resp. Corollaries 3.1 and 3.2 are automatically satisfied.

We shall assume that $f$ is increasing and convex on $I$, i.e. for all $x, y, z \in I$ we have $[x, y; f] > 0$. Let $[\alpha, \beta] \subset \operatorname{int}(I)$. Choose

$$g_1(x) = x - \tfrac{f(x)}{f_l'(\beta)} \quad \text{and} \quad g_2(x) = x - \tfrac{f(x)}{f_r'(\alpha)}$$

(the existence of the lateral derivatives $f_l'(\beta)$ and $f_r'(\alpha)$ is assumed by Proposition 1.4.). Obviously, $f_s'(\beta) > 0$ and $f_r'(\alpha) > 0$, since we have assumed that $f$ is increasing on $I$. From the assumption of convexity on $f$ it follows that $f$ is continuous on $[\alpha, \beta]$, and hence $g_1$ and $g_2$ are both continuous on $[\alpha, \beta]$, therefore satisfying hypothesis $(b)$. On the other hand, for all $x, y \in [\alpha, \beta]$ we have

$$[x, y; g_1] = 1 - \frac{1}{f_l'(\beta)} [x, y; f],$$

and since $f$ is convex we get that $[x,y;f] \leq f'_s(\beta)$, i.e. $[x,y;g_1] \geq 0$ (in other words, $g_1$ is an increasing function on $[\alpha,\beta]$).

A similar reasoning lead to the conclusion that $g_2$ is a decreasing function on $[\alpha,\beta]$.

Resuming, one can see that hypotheses (c) and (d) an both satisfied. The function $f$ is assumed to be increasing and so hypothesis (e) is verified. Hypothesis (f) is obviously satisfied from relation

$$0 < 1 - \frac{[x,y;f]}{f'_l(\beta)},$$

and from the fact that

$$0 < \frac{[x,y;f]}{f'_l(\beta)} < 1.$$

We choose now in (1.3) $x_0 = \alpha$ and we assume that $g_2(g_1(\alpha)) < \beta$, in which case the functions $f, g_1$ and $g_2$ satisfy in an obvious manner the hypotheses of Theorem 2.1.

**Remarks. 1.** From the above reasoning it follows that in order to obtain bilateral approximation sequences for the solution $\bar{x}$ of (1.1), there suffice monotonicity and convexity assumptions on $f$, followed by the condition $g_2(g_1(x_0)) \in I$.

**2.** If we choose $0 < \lambda_1 \leq f'_r(\alpha) < f'_l(\beta) \leq \lambda_2$, then the functions

$$\begin{aligned} g_1(x) &= x - \frac{f(x)}{\lambda_2} \\ g_2(x) &= x - \frac{f(x)}{\lambda_1}, \end{aligned}$$

obey conditions of Theorem 2.1.

**3.** If we choose the functions $g_1, g_2$ given by

$$\begin{aligned} g_1(x) &= x \\ g_2(x) &= x - \frac{f(x)}{\lambda_1} = g(x), \end{aligned}$$

then the hypotheses of Corollary 3.1 are fulfilled.

# Bibliography

[1] M. Balázs, *A bilateral approximating method for finding the real roots of real equations*, Rev. Anal. Numér. Théor. Approx., (21) **2** (1992), pp. 111–117.

[2] V. Casulli, D. Trigiante, *The convergence order for iterative multipoint procedures*, Calcolo, (13) **1** (1977), pp. 25–44.

[3] S. Cobzaş, *Mathematical Analysis*, Presa Universitară Clujeană, Cluj-Napoca (1997) (in Romanian).

[4] A.M. Ostrowski, *Solution of Equations and Systems of Equations*, (1960), Academic Press, New York and London.

[5] I. Păvăloiu, *On the monotonicity of the sequences of approximations obtained by Steffensens's method*, Mathematica (Cluj), (35) (58) **1** (1993), pp. 71–76.

[6] I. Păvăloiu, *Bilateral approximations for the solutions of scalar equations*, Rev. Anal. Numér. Théor. Approx., (23) **1** (1994), pp. 95–100.

[7] I. Păvăloiu, *Approximation of the roots of equations by Aitken-Steffensen-type monotonic sequences*, Calcolo, Vol.32, N°°1 – 2, 1995, pp.69–82.

[8] F.J. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J. (1964).

Received, 15 oct. 1999

North University of Baia Mare
Department of Mathematics and Computer Science,
Victoriei 76, 4800 Baia Mare
Romania