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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

FURTHER SOLUTIONS OF FALKNER-SKAN EQUATION

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Abstract. The nonclassical transformation is employed to derive the similarity differential equation

$|1+\lambda| f''' + ff'' + \beta(1-f'^2) = 0$, which governs the fluid flow past a moving wedge with included angle $\pi\beta$, and the parameter λ is the ratio of boundary velocity to the free stream velocity. In this research the behavior of the solution as $\lambda \to -1$ in investigated.

1. INTRODUCTION

The standard Falkner-Skan equation is given by

$$f'''(\eta) + f(\eta)f''(\eta) + \beta(1 - f'^2(\eta)) = 0 \quad (1)$$

with initial and boundary conditions

$$f(0) = \gamma, \quad f'(0) = -\lambda, \quad \text{and} \quad f'(\infty) = 1 \quad (2)$$

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This equation is related to the flow of an incompressible fluid over a moving wedge whose included angle is $\pi\beta$. The function $f(\eta)$ is the nondimensional stream function, η is the similarity ordinate and λ is the ratio of boundary velocity to the free stream velocity. This equation received an intensive research by many authors. In [1], Yang and Chien establish an analytic solution when $\lambda = 0$ and $\beta = -1$ using confluent hypergeometric functions, and obtain two types of unique analytic solution. The two types obtained are due to the different choice of the initial condition $f''(0)$. Hasting [2], studied the case $\lambda = 0$ and $\gamma = 0$ and proved that there exists one solution such that $f''(0) < 0$. Recently Riley and Weidman [3] studied the case $\gamma = 0$. They employed numerical calculations to study the existence and nonuniqueness of solution for $|\beta| \leq 1$ over a range of positive and negative values of λ. Their results indicate that for $-1 \leq \beta \leq 0$, two solutions exist for λ less than a critical value $\lambda_m(\beta)$ and no solution exists above $\lambda_m(\beta)$. They also observe that triple solutions exists for $0 < \beta \leq 0{,}14$.

In the present article we use the modified similarity transformation presented in [4], to derive the similarity differential equation that governs the flow. This will be accomplished in section two. In section three, discussion of the results restricted to the case $\gamma = 0$ will be presented.

2. MATHEMATICAL FORMULATION

The dimensional Navier-Stokes equations that describe the flow phenomenon are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4)$$

subject to the boundary conditions

$$u(x,0) = -\lambda U(x)\ ,\ v(x,0) = 0 \quad \text{and as} \quad y \to \infty\ ,\ u \to U_\infty \tag{5}$$

where u is the component of the velocity in the direction of the fluid flow, and v is the velocity in the direction normal to u. The constant ν is the kinematic viscosity and $U(x) = U_\infty x^m$, where m is related to the constant β by the relation

$$\beta = \frac{2m}{m+1}$$

To make the equations dimensionless, we follow the procedures presented in [4], and introduce the dimensionless coordinate η such for $\lambda \neq -1$

$$\eta = y\sqrt{\frac{m+1}{2}\,\frac{|1+\lambda|\,U_\infty}{\nu}\,x^{m-1}}\ .$$

The equation of continuity (4) can be integrated by introducing a stream function $\psi(x,y)$ given by

$$\psi(x,y) = \sqrt{\frac{2}{m+1}}\sqrt{\frac{\nu\, x\, U_\infty}{|1+\lambda|}\,x^{m+1}}\,f(\eta)$$

where $f(\eta)$ denotes the dimensionless stream function. Thus the velocity components become

$$u = \frac{\partial \psi}{\partial y} = U_\infty f'(\eta)\ ,$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{\frac{m+1}{2}}\sqrt{U_\infty\,\nu\,\frac{x^{m-1}}{|1+\lambda|}}\left\{f + \frac{m-1}{m+1}\eta f'\right\}$$

writing down the further terms of equation (3) then after simplification the following ordinary differential equation will result

$$|1+\lambda| f''' + ff'' + \beta(1 - f'^2) = 0 \tag{6}$$

with the boundary conditions

$$f(0) = 0\,,\quad f'(0) = -\lambda\,,\quad f'(\infty) = 1 \tag{7}$$

Note that when $\lambda = 0$, the problem is reduced to the standard Falkner-Skan equation (1) with $\gamma = 0$.

3. DISCUSSION OF THE RESULTS

Note at first that when $\lambda = -1$ one can show with the help of L'Hopital's rule that

$$\lim_{\lambda \to -1} \psi(x,y) = yU(x)f'(0) = yU(x)$$

furthermore

$$u|_{\lambda=-1} = \lim_{\lambda \to -1} \frac{\partial \psi}{\partial y} = U(x)f'(0) = U(x)$$

and

$$v|_{\lambda=-1} = \lim_{\lambda \to -1} -\frac{\partial \psi}{\partial x} = -yU'(x)$$

It is known that far away from the boundary layer the velocity component parallel to the stream, U(x), is simply U_∞, therefore, the stream function as well as the velocity components given by the above limits have a full physical meaning.

In the coming discussion we will restrict ourselves to the case $\lambda \to -1$. Therefore, in the neighborhood of $\lambda = -1$ say for $\lambda = -1 \pm \varepsilon$, where $0 < \varepsilon \ll 1$, the differential equation (10) could be written as

$$\varepsilon f''' + ff'' + \beta(1 - f'^2) = 0 \tag{8}$$

The above singular perturbation problem Equation (11) has no boundary layer when $\varepsilon \to 0$. The intensive numerical investigations suggested that the following form is suitable for the outer solution

$$f(\eta) = f_0 + f_1\,\varepsilon + f_2\,\varepsilon^2 + 0\,(\varepsilon^3) \tag{9}$$

substituting Equation (9) in Equation (8), and equating similar powers of ε, implies that f_0 satisfies the differential equation

$$f_0 f_0'' + \beta\,(1 - f_0'^2) = 0 \tag{10}$$

with the initial conditions

$$f_0(0) = 0\,, \quad f_0'(0) = 1\,, \quad \text{and} \quad f_0'(\infty) = 1 \tag{11}$$

Equation (10) has an exact solution which can be obtained using the standard techniques. It can be written as

$$\frac{f_0''}{1 - f_0'^2} = \frac{-\beta}{f_0} \tag{12}$$

and the first integral of equation (10) gives

$$\frac{1 - f_0'}{1 + f_0'} = A \exp\left(\int_{\varepsilon}^{\eta} \frac{\beta}{f_0(\zeta)}\, d(\zeta) \right)$$

where $\varepsilon > 0$ is chosen such that the integral in the right hand side converges, and a is a constant to be determined. Using the boundary condition at infinity gives the value of $A=0$, therefore equation (10) is reduced to $f_0'(\eta) = 1$ with the solution $f_0(\eta) = B\,\eta + C$. The initial condition at 0 gives the exact solution $f_0(\eta) = \eta$. Therefore the outer solution is given by

$$f_0(\eta) = \eta + 0\,(\varepsilon) \tag{14}$$

For the inner solution the following transformation proved to be adequate:

$$f(\eta) = \eta + \varepsilon^2\, g(\xi) \tag{15}$$

where $\xi = \frac{\eta}{\varepsilon}$.

The function g then satisfies the following differential equation:

$$g^{///} + [\varepsilon \xi + \varepsilon^2 g] g^{//} + \beta [1 - (1 + \varepsilon g^{/})^2] = 0 \qquad (16)$$

now assuming that

$$g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + ... \qquad (17)$$

and substituting Equation (17) in Equation (16) and equating equal powers of ε leads to the following form for the inner solution:

$$f_i(\eta) = (1 + \varepsilon)\eta + O(\varepsilon^3)$$

Figure 1 shows the solution curves for $f^{/}(\eta)$ when the two values of $\varepsilon = \pm 0,02$ are used for $\beta = 0,05$ and $\beta = -0,19884$. The last choice refers to the case of separated flow while the first choice is an arbitrary one. This figure draws a comparison between the present model given by Equation (4) and the model of Riley and Weidman [3] given by equation (1) with $\gamma = 0$. It indicates that for the present results, the solution converges to the asymptotic value $f^{/} = 1$ faster than the old model which implies that the boundary layer thickness is over estimated when the old model is used. However, several numerical experiments with various values of β indicate that β plays no significant role when ε is very small as shown in figure 2.

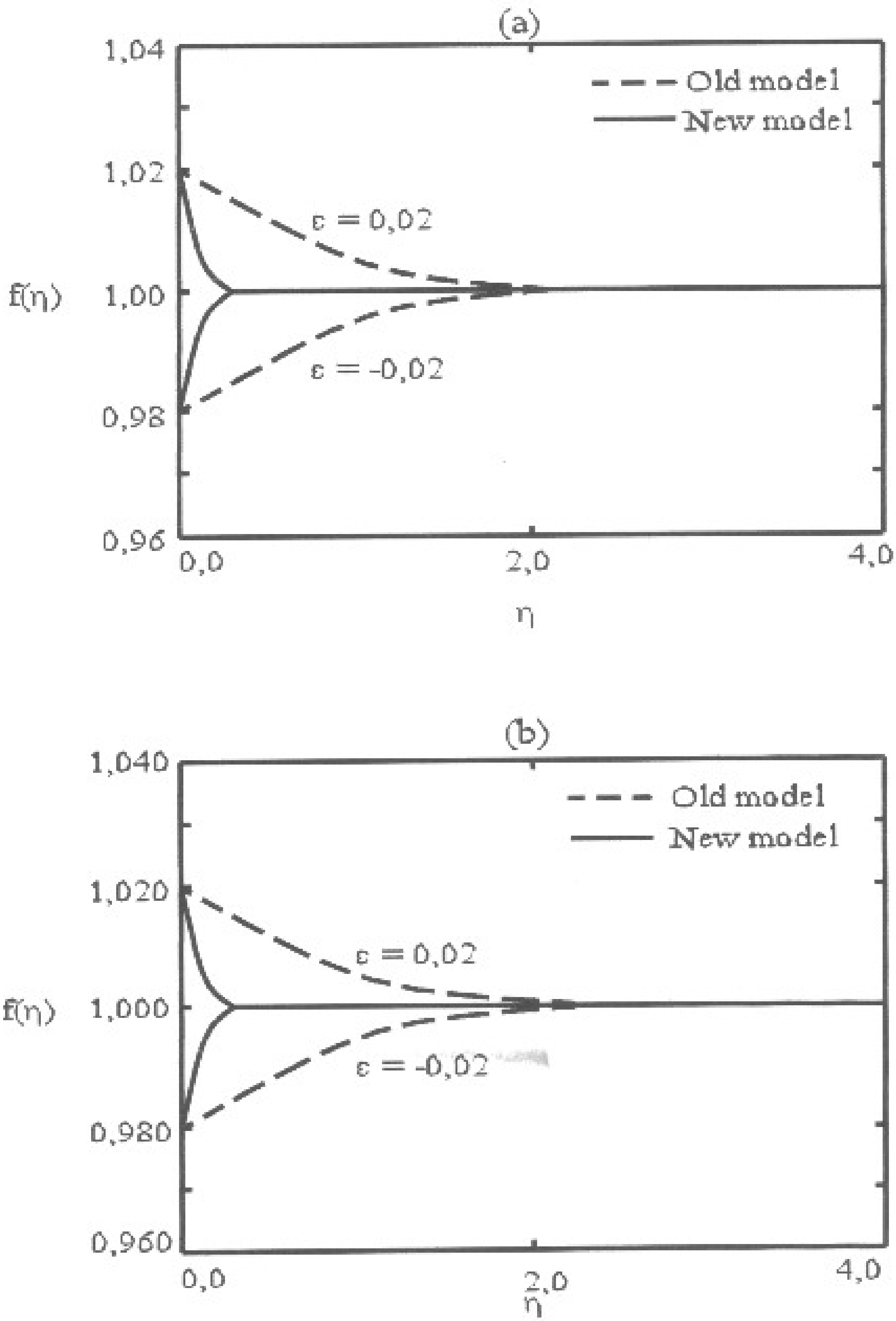

Figure 1:The velocity component $f'(\eta)$ for (a) β=0.05 and (b) β=−0.19884

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