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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

THE APPROXIMATION OF BIVARIATE FUNCTIONS BY USING GENERALIZED BERNSTEIN OPERATORS

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Abstract. The paper summarizes recent results obtained by the author, concerning the approximation of bivariate functions using Bernstein generalized operators.

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1. Preliminaries.

George M. Phillips ([9], [10], [11]) constructed a generalization of the classical operator of Bernstein in which the approximated function f is evaluated at intervals which are in geometric progression. We start by recalling some results due to G.M. Phillips containing q-integers.

Let $q > 0$ be any positive real number. For any non-negative integer i was introduced the so called "q-integer", denoted by [i] and defined by

$$[i] = \begin{cases} \dfrac{1-q^i}{1-q}, & q \neq 1 \\ i, & q = 1 \end{cases} \tag{1.1}$$

In an obvious way were defined a "q-factorial"

$$[i]! = \begin{cases} [i]\cdot[i-1]\cdots[1], & i = 1,2,\ldots \\ 1, & i = 0 \end{cases} \tag{1.2}$$

and a "q-binomial coefficient"

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{[k]!}{[r]!\,[k-r]!} \tag{1.3}$$

Next, for any positive integer n was constructed the sequence of positive linear operators $B_n : C[0,1] \to C[0,1]$, which associates to any $f \in C[0,1]$ the polynomial

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x) \tag{1.4}$$

In (1.4) an empty product denotes 1 and $f_r = f\left(\frac{[r]}{[n]}\right)$.

Clearly, if $q = 1$ the operator (1.4) reduces to the classical Bernstein operator.

It was proved ([9], [10], [11]) that the generalized Bernstein operator (1.4) has approximation properties similar to the properties of the classical Bernstein operator. In the present paper some extensions of the operator (1.4) to the case of bivariate functions will be presented.

2. The parametric extensions of the generalized Bernstein operator

Let $I^2 = [0,1] \times [0,1]$ be the unity square and let $\mathbb{R}^{I^2} = \{ f \mid f : I^2 \to \mathbb{R} \}$ be the space of real bivariate functions defined on this square. For any function $f \in \mathbb{R}^{I^2}$ and any positive real numbers $q_1, q_2 > 0$ let us denote

$$B_{n_1}^x(f;x,y) = \sum_{r_1=0}^{n_1} f_{r_1} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} x^{r_1} \prod_{s_1=0}^{n_1-r_1-1} (1-q_1^{s_1} x) \tag{2.1}$$

and

$$B_{n_2}^y(f;x,y) = \sum_{r_2=0}^{n_2} f_{r_2} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} y^{r_2} \prod_{s_2=0}^{n_2-r_2-1} (1-q_2^{s_2} y) \tag{2.2}$$

the parametric extensions of the operator (1.4). Note that in (2.1) and (2.2) an empty product denotes 1 and $f_{r_1} = f\left(\frac{[r_1]}{[n_1]}, y\right)$, $f_{r_2} = f\left(x, \frac{[r_2]}{[n_2]}\right)$.

Lemma 2.1. ([3]) The parametric extensions (2.1) and (2.2) of the operator (1.4) are linear positive operators on $C(I^2)$.

Lemma 2.2. ([3]) Let $f \in \mathbb{R}^{I^2}$. Then, for any $y \in [0,1]$ the operator (2.1) has the following interpolation properties:

(i) $B_{n_1}^{x}(f;0,y) = f(0,y)$; (ii) $B_{n_1}^{x}(f;1,y) = f(1,y)$.

Lemma 2.3. ([3]) Let $f \in \mathbb{R}^{I^2}$. Then, for any $x \in [0,1]$ the operator (2.2) has the following interpolation properties:

(i) $B_{n_2}^{y}(f;x,0) = f(x,0)$; (ii) $B_{n_2}^{y}(f;x,1) = f(x,1)$.

Lemma 2.4. ([3]) The operators $B_{n_1}^{x}, B_{n_2}^{y}$ commute on $C(I^2)$. Their product is the linear positive operator $B_{n_1,n_2} : C(I^2) \to C(I^2)$, which associates to any function $f \in C(I^2)$ the approximant

$$B_{n_1,n_2}(f;x,y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} f_{r_1,r_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1} y^{r_2} \prod_{s_1=0}^{n_1-r_1-1} (1 - q_1^{s_1} x) \prod_{s_2=0}^{n_2-r_2-1} (1 - q_2^{s_2} y) \quad (2.3)$$

Lemma 2.5. ([3]) The generalized bivariate Bernstein operator (2.3) interpolates the function f in the four cornes of the unity square, i. e.

$$B_{n_1,n_2}(f;0,0) = f(0,0) ; \quad B_{n_1,n_2}(f;0,1) = f(0,1) ;$$

$$B_{n_1,n_2}(f;1,0) = f(1,0) ; \quad B_{n_1,n_2}(f;1,1) = f(1,1) .$$

Lemma 2.6.([3])Let $e_{ij} : I^2 \to I^2$, $e_{ij}(x,y) = x^i y^j$, $(0 \le i+j \le 2$, *i,j-integers*) be the test functions. Then, the following equalities

(i) $B_{n_1,n_2}(e_{00};x,y) = e_{00}(x,y)$;

(ii) $B_{n_1,n_2}(e_{10};x,y) = e_{10}(x,y)$;

(iii) $B_{n_1,n_2}(e_{01};x,y) = e_{01}(x,y)$;

(iv) $B_{n_1,n_2}(e_{11};x,y) = e_{11}(x,y)$;

(v) $B_{n_1,n_2}(e_{20};x,y) = e_{20}(x,y) + \dfrac{x(1-x)}{[n_1]}$;

(vi) $B_{n_1,n_2}(e_{02};x,y) = e_{02}(x,y) + \dfrac{y(1-y)}{[n_2]}$,

hold, for any $(x,y) \in I^2$.

3. The approximation of the continuous functions using the generalized bivariate Bernstein operators

Theorem 3.1. *([3]) Let be* $q_1 = q_1(n_1)$, $q_2 = q_2(n_2)$ *and let* $q_1(n_1) \to 1$, $q_2(n_2) \to 1$ *from below as* $n_1 \to \infty$, $n_2 \to \infty$. *Then, for any* $f \in C(I^2)$ *the sequence of bivariate generalized Bernstein operators defined by (2.3), converges uniformly to* $f(x,y)$ *on* I^2 .

Theorem 3.2. *([3]) For any* $f \in \mathbb{R}^{I^2}$, *bounded on* I^2, *the inequality*

$$\| f - B_{n_1,n_2}(f) \|_\infty \le \frac{9}{4}\, \omega\left(\frac{1}{\sqrt{[n_1]}}, \frac{1}{\sqrt{[n_2]}} \right) \qquad (3.1)$$

holds. In (3.1), $\| \cdot \|_\infty$ *denotes the uniform norm and* ω *denotes the first order modulus of smoothness.*

Next, some particular cases will be discussed.

First, one suppose that $q_1 = 1$. In this case, we obtain the operator

$$B_{n_1,n_2}(f;x,y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} f_{r_1,r_2} \binom{n_1}{r_1} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1} (1-x)^{n_1-r_1} \cdot y^{r_2} \prod_{s_2=0}^{n_2-r_2-1} (1-q_2^{s_2} y) \quad (3.2)$$

Note that in (3.2) an empty product denotes 1 and $f_{r_1,r_2} = f\left(\frac{r_1}{n_1}, \frac{[r_2]}{[n_2]} \right)$.

As a consequence of the theorem 3.1, it follows

Corollary 3.1. Let be $q_2 = q_2(n_2)$ and let $q_2(n_2) \to 1$ as $n_2 \to \infty$.

Then, for any $f \in C(I^2)$ the sequence of generalized Bernstein operators defined at (3.2) converges uniformly to $f(x,y)$ on I^2.

As a consequence of the theorem 3.2, it follows

Corollary 3.2. For any $f \in \mathbb{R}^{I^2}$, bounded on I^2, the inequality

$$\| f - B_{n_1,n_2}(f) \|_\infty \le \frac{9}{4}\,\omega\left(\frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{[n_2]}} \right)$$

holds.

Next, one suppose that $q_2 = 1$. In this case, we obtain the following generalized bivariate Bernstein operator

$$B_{n_1,n_2}(f;x,y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} f_{r_1,r_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \binom{n_2}{r_2} x^{r_1} \cdot y^{r_2} (1-y)^{n_2-r_2} \prod_{s_1=0}^{n_1-r_1-1} (1 - q_1^{s_1} x) \quad (3.4)$$

In (3.4) an empty product denotes 1 and $f_{r_1,r_2} = f\left(\frac{[r_1]}{[n_1]}, \frac{r_2}{n_2} \right)$.

As consequence of the theorems 3.1 and 3.2, one obtain

Corollary 3.3. Let be $q_1 = q_1(n_1)$ and let $q_1(n_1) \to 1$ as $n_1 \to \infty$.

Then, for any $f \in C(I^2)$ the sequence of generalized Bernstein operators defined at (3.4) converges uniformly to $f(x,y)$ on I^2.

Corollary 3.4. For any $f \in \mathbb{R}^{I^2}$, bounded on I^2, the inequality

$$\| f - B_{n_1,n_2}(f) \|_\infty \le \frac{9}{4}\,\omega\left(\frac{1}{\sqrt{[n_1]}}, \frac{1}{\sqrt{n_2}} \right) \quad (3.5)$$

holds.

Remark 3.1. For $q_1 = q_2 = 1$, the operator (2.3) reduces to the classical bivariate Bernstein operator [12].

4. Bivariate generalized Bernstein operators of blending type

Definition 4.1. [7] Let $B_{n_1}^x, B_{n_2}^y$ be the parametric extensions of the generalized Bernstein operator (1.4). The boolean sum operator $B_{n_1}^x \oplus B_{n_2}^y : \mathbb{R}^{I^2} \to \mathbb{R}^{I^2}$, defined by

$$B_{n_1}^x \oplus B_{n_2}^y = B_{n_1}^x + B_{n_2}^y - B_{n_1}^x B_{n_2}^y \tag{4.1}$$

is called generalized bivariate Bernstein operator of blending type.

Remark 4.1. [4] The generalized bivariate Bernstein operator of blending type associates to any $f \in \mathbb{R}^{I^2}$ the approximant

$$\left(B_{n_1}^x \oplus B_{n_2}^y\right)(f; x, y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1} y^{r_2} \prod_{s_1=0}^{n_1-r_1-1} (1 - q_1^{s_1} x) \cdot$$

$$\cdot \prod_{s_2=0}^{n_2-r_2-1} (1 - q_2^{s_2} y) \cdot (f_{r_1} + f_{r_2} - f_{r_1} f_{r_2}) \tag{4.2}$$

In (4.2) an empty product denotes 1, $f_{r_1} = f\left(\frac{[r_1]}{[n_1]}, y\right)$, $f_{r_2} = f\left(x, \frac{[r_2]}{[n_2]}\right)$,

$f_{r_1, r_2} = f\left(\frac{[r_1]}{[n_1]}, \frac{[r_2]}{[n_2]}\right)$.

Note that (4.2) is a pseudopolynomial in the Marchaud Sense [1].

Remark 4.2. For $q_1 = q_2 = 1$, from (4.2) one obtains the classical bivariate Bernstein operator of blending type, first studied by E. Dobrescu and I. Matei [3].

Using the expression (4.2), it follws by direct computation

Theorem 4.1. *[4] The generalized bivariate Bernstein operator of blending type* $B_{n_1}^x \oplus B_{n_2}^y$ *interpolates the function* $f \in \mathbb{R}^{I^2}$ *on the boundary of the unity square, i.e.*

$$(B_{n_1}^x \oplus B_{n_2}^y)(f;0,y) = f(0,y)\ ;\quad (B_{n_1}^x \oplus B_{n_2}^y)(f;1,y) = f(1,y)$$

$$(B_{n_1}^x \oplus B_{n_2}^y)(f;x,0) = f(x,0)\ ;\quad (B_{n_1}^x \oplus B_{n_2}^y)(f;x,1) = f(x,1)$$

for any $(x,y) \in I^2$.

5. The approximation of B-continuous functions using the generalized bivariate Bernstein operators of blending type

The notions of B-continuous function, B-bounded function and the relationships between these notions were introduced by K. Bögel [6]. A. Korovkin type theoreme for the approximation of the B-continuous functions was established by C. Badea, I. Badea and H.H. Gonska [1].

Theorem 5.1.[4] Let $q_1 = q_1(n_1)$, $q_2 = q_2(n_2)$ and let $q_1(n_1) \to 1$, $q_2(n_2) \to 1$ from below as $n_1 \to \infty$, $n_2 \to \infty$. Then, for any $f \in C_b(I^2)$ the sequence $\{(B_{n_1}^x \oplus B_{n_2}^y)(f)\}$ converges to f, uniformly on I^2.

Remark 5.1. In the theoreme 5.1, $C_b(I^2)$ denotes the set of all B-continuous functions on I^2.

Theorem 5.2. *[4] For any* $f \in \mathbb{R}^{I^2}$, *B-bounded on* I^2*, the inequality*

$$\| f - (B_{n_1}^x \oplus B_{n_2}^y)(f) \|_\infty < \frac{q}{4}\, \omega_{mixed}\left(\frac{1}{\sqrt{[n_1]}}, \frac{1}{\sqrt{[n_2]}} \right) \qquad (5.1)$$

holds. In (5.1), $\|\cdot\|_\infty$ *denotes the uniform norm and* ω_{mixed} *is the mixed modulus of smoothness [1].*

Next, one suppose that $q_1 = 1$. In this case, one obtains the approximant

$$\left(B_{n_1}^{x} \oplus B_{n_2}^{y}\right)(f;x,y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \binom{n_1}{r_1} \begin{bmatrix} n_2 \\ r_2 \end{bmatrix} x^{r_1}(1-x)^{n_1-r_1} y^{r_2} \prod_{s_2=0}^{n_2-r_2-1} (1-q_2^{s_2}y)\cdot$$

$$\cdot\,(f_{r_1}+f_{r_2}-f_{r_1}f_{r_2}) \tag{5.2}$$

In (5.2) an empty product denotes 1, $f_{r_1}=f\left(\frac{r_1}{n_1},y\right)$, $f_{r_2}=f\left(x,\frac{[r_2]}{[n_2]}\right)$, $f_{r_1r_2}=f\left(\frac{r_1}{n_1},\frac{[r_2]}{[n_2]}\right)$.

As consequences of the theoremes 5.1 and 5.2, it follows

Corollary 5.1. Let $q_2=q_2(n_2)$ and let $q_2(n_2)\to 1$ as $n_2\to\infty$. Then, for any $f\in C_b(I^2)$, the sequence $\left\{\left(B_{n_1}^{x}\oplus B_{n_2}^{y}\right)(f)\right\}$ defined at (5.2), converges to f, uniformly on I^2.

Corollary 5.2. For any $f\in\mathbb{R}^{I^2}$, B-bounded on I^2, the inequality

$$\left\|f-\left(B_{n_1}^{x}\oplus B_{n_2}^{y}\right)(f)\right\|_\infty \le \frac{9}{4}\,\omega_{mixed}\left(\frac{1}{\sqrt{n_1}},\frac{1}{\sqrt{[n_2]}}\right) \tag{5.3}$$

holds.

If one suppose that $q_2=1$, one obtains the approximant

$$\left(B_{n_1}^{x} \oplus B_{n_2}^{y}\right)(f;x,y) = \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \begin{bmatrix} n_1 \\ r_1 \end{bmatrix} \binom{n_2}{r_2} x^{r_1} y^{r_2}(1-y)^{n_2-r_2-1} \prod_{s_1=0}^{n_1-r_1-1} (1-q_1^{s_1}x)\cdot$$

$$\cdot\,(f_{r_1}+f_{r_2}-f_{r_1}f_{r_2}) \tag{5.4}$$

By applying Theorems 5.1 and 5.2, it follows

Corollary 5.3. Let $q_1=q_1(n_1)$ and let $q_1(n_1)\to 1$ as $n_1\to\infty$. Then, for any $f\in C_b(I^2)$, the sequence $\left\{\left(B_{n_1}^{x}\oplus B_{n_2}^{y}\right)(f)\right\}$ defined (5.4), converges to f, uniformly on I^2.

Corollary 5.4. For any $f \in \mathbb{R}^{I^2}$, B-bounded on I^2, the inequality

$$\left\| f - \left(B_{n_1}^{x} \oplus B_{n_2}^{y} \right)(f) \right\|_{\infty} \le \frac{9}{4} \omega_{mixed} \left(\frac{1}{\sqrt{[n_1]}}, \frac{1}{\sqrt{n_2}} \right) \quad (5.5)$$

holds.

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