Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XV(1999), Nr. 1-2, 27-32

Dedicated to Professor Ion PĂVĂLOIU on his 60^{th} anniversary

FIXED POINT THEOREMS FOR NONEXPANSIVE OPERATORS ON NONCONVEX SETS

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Abstract. Two theorems on the existence of fixed points of nonexpansive selfmappings of a nonconvex set are proved. The results in the present paper extend the corresponding theorems in [3].

Let E be a Banach space, C a subset of E and T a selfoperator of C. It is well-known (see [2], for example) that in an uniformly convex Banach space every nonexpansive operator T of a closed bounded convex subset C of E has at least one fixed point in C.

(We recall that $T: C \to C$ is nonexpansive if for al x,y in C $\|Tx - Ty\| \le \|x - y\|$)

Dotson [3] proved similar results for (weakly) compact subsets S of a Banach space, when the convexity of S is replaced by some other properties, described by means of a family of functions from [0,1] into S. The aim of this paper is to extend the Dotson's results, by considering a generalized contractive condition instead of that given in [3].

DEFINITION 1, ([1], [5]). A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called **comparison function** if

(i) φ is monotone increasing;

(ii) the sequence $\{\varphi^n(r)\}_{n\geq 0}$ converges to 0, for each $r \in \mathbb{R}_+$

(φ^n stands for the n^{th} iterate of φ).

EXAMPLE 1. If $0<t<1$, then $\varphi_t(r) = t\cdot r$, for each $r \in \mathbb{R}_+$, is a typical comparison function. There exist non-continuous and nonlinear comparison functions (see [1]).

DEFINITION 2. Let S be a subset of the Banach space E, and let $F=\{f_\alpha\}_{\alpha\in S}$ be a family of functions from [0,1] into S, having the property that for each $\alpha \in S$ we have $f_\alpha(1)=\alpha$. Such a family is said to be **φ-contractive** provided that, for all α and β in S and for all t in (0,1) there exists a comparison function φ_t such that

$$\| f_\alpha(t) - f_\beta(t) \| \le \varphi_t(\| \alpha - \beta \|)$$

Such a family F is said to be **jointly continuous** provided that if $t \to t_0$ in [0,1] and $\alpha \to \alpha_0$ in S then

$$f_\alpha(t) \to f_{\alpha_0}(t_0) \quad \text{in S.}$$

THEOREM 1. *Suppose S is a compact subset of a Banach space E, and suppose there exists a φ - contractive and jointly continuous family of functions associated with S as in Definition 2. Then any nonexpansive selfoperator T of S has a fixed point in S.*

Proof. Let $\{k_n\}_{n\ge 1}$ be a sequence of numbers, $0<k_n<1$ with $\lim_{n\to\infty} k_n = 1$ and let $T_n : S \to S$ be defined by

$$T_n x = f_{Tx}(k_n), \quad \text{for all } x \in S.$$

Since $T(S)\subseteq S$, each T_n is well-defined and maps S into S. Furthermore, for each *n* and for all *x*, *y* in S we have

$$\| T_n x - T_n y \| = \| f_{Tx}(k_n) - f_{Ty}(k_n) \| \le \varphi_t(\| Tx - Ty \|) \le \varphi_t(\| x - y \|),$$

since φ is monotone increasing and T is nonexpansive. This shows that, for each n, T_n is a φ - contraction (see [1], [5]). On the other hand, as a compact (hence closed) subset of the Banach space E, S is a complete metric space. Therefore, by the generalized contraction mapping principle (see [1], Theorem 1.5.1 or [5]), each operator T_n has an unique fixed point $x_n \in S$.

Since S is compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to$ some $x \in S$. But $T_{n_j} x_{n_j} = x_{n_j}$, hence

$$Tx_{n_j} \to x \, , \quad \text{as} \quad j \to \infty \, .$$

Since T is continuous (as a nonexpansive operator), it results that

$$Tx_{n_j} \to Tx \, , \quad \text{as} \quad j \to \infty$$

which together with the joint continuity yields

$$T_{n_j} x_{n_j} = f_{T_{n_j}}(k_n) \to f_{Tx}(1) = Tx \, .$$

As E is Haussdorf, it follows that $Tx = x$, that is, T has a fixed point in S. The proof is complete.

Remarks. 1) For $\varphi_t(r) = \phi(t) \cdot r$, $r \in \mathbb{R}_+$ and $\phi(t) \in (0,1)$, from Theorem 1 we obtain Theorem 1 of [3].

2) a special case of the above case is Theorem 1 of [4], where S is assumed to be star-shaped. With p a star-centre and $k_n = n/(n+1)$ we have $f_\alpha(t) = (1-t)p + t\alpha$ so that

$$T_n x = f_{Tx}(k_n) = (1-k_n)p + k_n TX \, ,$$

and one easily checks that, in this case, φ_t is as in Example 1:

$$\| f_\alpha(t) - f_\beta(t) \| \le t \, \| \alpha - \beta \| \, ,$$

and that $f_\alpha(t)$ is jointly continuous with respect to t and α.

DEFINITION 3. A family $F = \{f_\alpha\}_{\alpha \in S}$ of functions from [0,1] into a set S is said to be **jointly weakly continuous** provided that if $t \to t_0$ in [0,1] and $\alpha \to \alpha_0$ in S then $f_\alpha(t) \rightharpoonup f_{\alpha_0}(t_0)$ in S ($\rightharpoonup$ denotes weak convergence).

THEOREM 2. *Let S be a weakly compact subset of a Banach space E and suppose there exists a φ - contractive, jointly weakly continuous family F of functions associated with S as in Definition 1. Then any nonexpansive weakly continuous selfoperator T of S has a fixed point in S.*

Proof. We repeat mainly the arguments in [3]. For $\{k_n\}$ as in the proof of the Theorem 1, we define $T_n : S \to S$ by $T_n x = f_{Tx}(k_n)$ for all $x \in S$ and for all $n = 1, 2, 3, \ldots$. Then, each T_n is a φ - contraction on S. Since the weak topology of E is Haussdorf and S is weakly compact, it results that S is weakly closed and therefore strongly closed. Hence S is a complet metric space with respect to the norm topology of the Banach space E, and so each T_n has an unique fixed point $x_n \in S$. By the Eberline-Smulian theorem, S is weakly sequentially compact. Thus there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to$ some $x \in S$. Since $T_{n_j} x_{n_j} = x_{n_j}$, it results $T_{n_j} x_{n_j} \to x$ and since T is weakly continuous, we have $Tx_{n_j} \to Tx$. The joint weak continuity now yields

$$T_{n_j} x_{n_j} = f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx}(1) = Tx$$

and since the weak topology is Haussdorf, we deduce that $Tx = x$, which ends the proof.

Remark. For $\varphi_t(r) = \phi(t) \cdot r$, $r \in \mathbb{R}_+$ and $\phi : (0,1) \to (0,1)$ a given function, from Theorem 2 in this paper we obtain Theorem 2 in [3].

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Received: 2.02.1999

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