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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

ON THE INEXACT UZAWA METHODS FOR SADDLE POINT PROBLEMS ARISING FROM CONTACT PROBLEM

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Abstract. In this paper we consider the so called "inexact Uzawa" algorithm for solving saddle point systems which arise in the discretization of contact problems. By using appropriate Lagrangeanes one can transform the original problem (the contact problem), into a saddle point problem on a convex set.

The inexact Uzawa methods replace the exact inverse of a matrix A by a "incomplete" or "approximate" evaluation of A^{-1}.

We discuss convergence and applications of inexact Uzawa methods to solving the contact problems.

1. INTRODUCTION

We consider the abstract Uzawa algorithm for linear saddle point problems:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{1.1}$$

where A is an $n \times n$ symmetric positive definite matrix, B is an $m \times n$ matrix, C is an $m \times m$ symmetric positive semidefinite matrix, p is a vector in $\mathbb{R}^n$, q is a vector in $\mathbb{R}^m$. For this problem Elman and Golub [4] given the convergence results.

Additional convergence results of inexact Uzawa methods for linear saddle point problems (1.1) were given by Bramble, Pasciak and Vassilev [1].

The preconditioned inexact Uzawa algorithm is defined as fellows [4].

Given an initial approximation y_0 of y

for k=0 until convergence, do

Compute x_{k+1} such that $Ax_{k+1} = p - B^T y_k + \delta_k$

Compute

$$y_{k+1} = y_k + \alpha Q^{-1}(Bx_{k+1} - Cy_k - q) \qquad (1.2)$$

enddo.

The vector δ_k is residual of the approximate solution x_{k-1} to the system $Ax = p - B^T y_k$, α is a positive stepsize, and Q is an $m \times m$ symmetric positive definite matrix.

A natural extension of method (1.2) to nonlinear saddle points problems, when $Ax = F(x)$, with $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone mapping with modulus μ, and F is not necessarily differentiable, is

Given an initial approximation y_0 of y
for k=0 until convergence, do

Compute x_{k+1} such that $F(x_{k+1}) = p - B^T y_k + \delta_k$.

Compute

$$y_{k+1} = y_k + \alpha_k Q_k^{-1}(Bx_{k+1} - Cy_k - q) \qquad (1.3)$$

enddo.

The vector δ_k is again the residual of the approximate solution x_{k+1} to the system $F(x) = p - B^T y_k$ α_k is a positive stepsise, and Q_k is an $m \times m$ symmetric positive definite matrix.

After finite element discreditation of the problems we obtain big sparse nonsymmetric ill-conditioned linear systems of equations. Solving these systems by direct methods (as Gaussian elimination e.g.) is not an efficient way (or even impossible sometime) because of too big computer memory requested for storage and very big number of elementary arithmetic operation which together with the ill-conditioning aspect can give rise to the computational errors. Classical iterative methods are also not indicated (or impossible to be used) for such kind of non-symmetric systems because of their bad convergence properties.

One way to overcome these difficulties is to use preconditioning of the system i.e. to transform it by (formally) left and/right matrix multiplications such the new matrix obtained has more "claustered" spectrum, thus a "small" (independent on the mesh size) condition number. This paper presents such a preconditioning technique, for Uzawa algorithm which uses incomplete decomposition of the Gram matrix of the finite element basis functions.

We apply these solvers to an unilateral contact problem with friction between several elastic bodies. The non-linearities are the inequalities associated with the unilateral conditions and with the friction law.

2. VARIATIONAL FORMULATION OF THE CONTACT PROBLEMS

The variational form of the contact problem with friction is the following variational inequality of the second kind:

Find $u \in V$ such that

$$b(u, v-u) + j_t(v) - j_t(u) \geq L(v-u) \quad \forall\ v \in V \tag{2.1}$$

with $b(u,v) = a(u,v) + j_n(u,v)$, *where:*

- $V = \{ v \in (H^1(\Omega))^d : v = 0 \ on\ \Gamma_D \}$ *is the function space;*

$\Omega \in \mathbb{R}^d$, $d=2$ *or* $d=3$ is an elastic body which occupies an open bounded Lipschitz domain, that come in contact with the rigid foundation;

- $\Gamma = \delta\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ is the Lipschitz continuous boundary of the Ω,
- $a(u,v) = \int_\Omega \sigma(u)\, \varepsilon(v)\, dx$ is the virtual work produced by the action of the stress σ on the strain ε;
- $j_n(u,v) = \int_{\Gamma_c} c_n (u_n - g)_+^{m_n} v_n\, ds$ is the virtual work produced by normal pressure on the contact boundary Γ_C;
- $j_t(u) = \int_{\Gamma_C} c_n p_T \mid v_t \mid ds$ is the virtual work produced by tangential pressure on the contact boundary Γ_C; with $p_T = (u_n - g)_+^{m_T}$, $g \geq 0$ is the initial gap between Γ_C and the fundation, j_t is the non-differentiable functional;
- $L(v) = \int_\Omega F v\, dx + \int_{\Gamma_N} T v\, ds$ is the virtual work produced by the volume

force F in Ω and by the surface traction T on Γ_N ;

- c_n, c_T, m_n, m_T are material constants depending on boundary contact properties, and $(\cdot)_+ = \max(\cdot, 0)$.

We denote the multiplier space:

$$\Lambda = \{ \lambda \in (L^2(\Gamma_C)^{d-1}) : |\lambda| \le 1 \ on \ supp\, p_T, \lambda = 0 \ on \ \Gamma_C - supp\, p_T \} \quad (2.2)$$

and the restrains set:

$$K = \{ v \in V ; \quad v_n \le g \} \ .$$

The existence of a multiplies for the problem (2.1) is given by:

Theorem 2.1. *The function u∈V is the solution of the problem (2.1) if and only if exists of λ, such that*

$$b(u,v) + \int_{\Gamma_C} c_T p_T \lambda v_T ds = L(v) , \quad \forall \ v \in V \quad (2.3)$$

$$\lambda \in \Lambda , \ \lambda v_T = | v_T | , \ on \ \Gamma_C \ . \quad (2.4)$$

For the proof see [6].

3. VARIATIONAL INEQUALITY FORMULATION AS A SADDLE POINT PROBLEM

Following procedure from Céa and Glowinski [2], we define a Lagrangean $\mathcal{L}$ an $K \times \Lambda$.

Theorems 3.1. *Let* $u \in K$, $\lambda \in \Lambda$ *satisfying (2.3) and (2.4). Then,* (u, λ) *is the unique saddle point of* $\mathcal{L}$ *on* $K \times \Lambda$ *i.e.*

$$\mathcal{L}(u,\eta) \le \mathcal{L}(u,\lambda) \le \mathcal{L}(v,\lambda) , \quad \forall \ v,\eta \in K \times \Lambda \ . \quad (2.5)$$

For the proof see [6].

A consequence of Theorem 2.1 is

Proposition 3.1. For the solution of the problem (2.1), there exists a unique $\lambda \in \Lambda$, with $\lambda v_T = | v_T |$, such that

$$b(u,v) = L(v) + \int_{\Gamma_C} c_T p_T \lambda v_T \, ds \,, \quad \forall \; v \in K \;.$$

Remarks. The relationship between the solution of problem (2.1), denoted by $\overline{u}$, on a hand, and the saddle point (u, λ) of the second inequality in (2.5), by the other hand, is given by

$$\overline{u} = u \in \Omega \,, \; \sigma_T(u) = -c_p p_T \lambda \;\; a.e. \;\; on \;\; \Gamma_C \;. \tag{2.6}$$

Therefore the contact problem with given friction can be approximated by solving the saddle point problem (2.5). This means a minimization problem which contains a non-differentiable functional has been replaced by an other one in which the Lagrangean is regular with respect to each variable. From (2.6) we can deduce a very important mechanical interpretation for the Lagrange's multipliers.

One other benefit of this formulation consists in the possibility of exploiting of an Uzawa-type algorithm for the solution of the problem (2.5), which has to simultaneously compute the displacements as well as the contact tangential stresses σ_T .

4. FINITE ELEMENT APPROXIMATIONS AND THE SOLUTION OF THE SYSTEM OF EQUATIONS DERIVED FROM A PERTURBED LAGRANGEAN

Using standard finite element procedures, approximate version of problem (2.1) can be constructed in finite-dimensional spaces $V_h (\subset V \subset V')$ resulting one discrete problem $(P_k)_h$. For a certain (h) the approximate displacements at each time t are elements of V_h

$$v^h \in V_h \tag{4.1}$$

within each element $\Omega_e^h \, (e = 1, ..., E_h)$. Every discrete problem $(P_k)_h$ is a static one, it requires approximate updating of the displacements and the loads after each increment.

For each static problem $(P_k)_h$ we consider an inter approximation $(V_h, K_h, j_{nh}, j_{th})$ and we formulate the following discrete problem:

Problem $(Pk)_h^i$. *Find* $u_h^i \in K_h$ *such that*

$$b(u_h^i, v_h - u_h^i) + j_{th}(u_h^i, v_h) - j_{th}(u_h^i, u_h^i) \geq L_h^i(v_h - v_h^i) - F^i(u_h^i, v_h - u_h^i) \quad (4.2)$$

We consider a discrete variational formulation of the incremental problem $(Pk)_h^i$, using for the contact area a three nodes contact element for the two dimensional case [8]. In the three dimensional case a four node contact element consisting of three "master" nodes and one "slave" node, is employed (see [7]).

In all numerical applications we derived a perturbed Lagrangean formulation for the case of frictional stick and for the case of frictional slide. For the case of frictional stick the perturbed Lagrangean for the bodies in contact has the following form:

$$\mathcal{L}_d(u, \Sigma_n, \Sigma_t, \Sigma_\tau) = \frac{1}{2} b(u,u) - L(u) + \quad (4.3)$$

$$+ \Sigma_n^T G_n + \Sigma_t^T G_t + \Sigma_\tau^T G_\tau - \frac{1}{2\omega_n} \Sigma_n^T \Sigma_n - \frac{1}{2\omega_t} \Sigma_t^T \Sigma_t - \frac{1}{2\omega_\tau} \Sigma_\tau^T \Sigma_\tau$$

where u is the vector nodal displacement $\sum_n, \sum_t, \sum_\tau$ are the vectors of normal and tangential nodal contact forces, respectively, G_n, G_t, G_τ are the vectors of normal and tangential nodal gaps and $\omega_n, \omega_t, \omega_\tau$ are the normal and tangential penalty parameters respectively.

The Newton-Raphson method was applied to the discrete variational formulations that can be derived from these perturbed Lagrangean functionals.

A standard assembly procedure can be used to add the contact contributions of each contact element to the global tangent stiffness and residual matrix and thus we obtain

$$KU = R \quad (4.4)$$

where

$$K = K_B + \sum_{s=1}^{S} K_C^s, \quad R = -\left(R_B + \sum_{s=1}^{S} R_C^s \right) \quad (4.5)$$

K_B, R_B are mechanical global tangent stiffness matrix and residual vector K_C^s, R_C^s

are mechanical contributions of contact nod s, $U = (\Delta U, \Delta \Sigma_n, \Delta \Sigma_t, \Delta \Sigma_\tau)^T$ is approximated solution, S is the total number of the slave nodes.

For the case of frictional slide the relation $|\Sigma_{tan}| = \mu |\Sigma_n|$, where μ is the coefficient of friction and $\sum_{tan}$ is the result force of the Σ_t and Σ_τ, forces in the tangent plane of the contact surface.

5. FORMULATION AND ALGORITHM FOR INEXACT UZAWA SADDLE POINT PROBLEMS ARISING FROM CONTACT PROBLEM

The perturbed Lagrangean (4.3) is equivalent with a saddle point problem:

find $(U, \Sigma_n, \Sigma_t, \Sigma_\tau)$ *s.t.*

$$\mathfrak{L}_d(U, \Sigma_n^1, \Sigma_t^1, \Sigma_\tau^1) \le \mathfrak{L}_d(U, \Sigma_n, \Sigma_t, \Sigma_\tau) \le \mathfrak{L}_d(V, \Sigma_n, \Sigma_t, \Sigma_\tau) \qquad (5.1)$$

$$\forall \ (V, \Sigma_n, \Sigma_t, \Sigma_\tau) \in V_h \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \ .$$

If we note with $p = (\Sigma_n, \Sigma_t, \Sigma_\tau)$, the solution of (4.3) is equivalent with the solution of the algebric equations:

$$\frac{\partial \mathfrak{L}_d(U,p)}{\partial U} = 0 \ , \quad \frac{\partial \mathfrak{L}_d(U,p)}{\partial p} = 0 \ . \qquad (5.2)$$

The matriceal form is:

$$\begin{pmatrix} K_B & B^T \\ B & K_C \end{pmatrix} \begin{pmatrix} \Delta U \\ \Delta p \end{pmatrix} = \begin{pmatrix} R_B \\ R_C \end{pmatrix} \ . \qquad (5.3)$$

Starting with an initial approximation Δp_0 of Δp, Uzawa algorithm constructs a sequence of approximations ΔU_k and Δp_k as follows:

for k=0 until converge, do

Solve $K_B \Delta U_{k+1} = R_B - B^T \Delta p_k$

Compute $\Delta p_{k-1} = \Delta p_k + \alpha (B \Delta U_{k+1} + K_C \cdot \Delta p_k - R_C)$

enddo.

Elimination of ΔU_{k+1} from the construction of Δp_{k-1} gives the iteration:

$$\Delta p_{k-1} = \Delta p_k + \alpha\left[B K_B^{-1}\cdot R_B - R_C - \left(B\cdot K_B^{-1} B^T - K_C\right)\cdot \Delta p_k\right] \qquad (5.4)$$

for the unknowns $\{\Delta p_k\}$. This is a fixed-parameter first-order Richardson applied to the system

$$\left[B\cdot K_B^{-1} B^T - K_C\right][\Delta p_k] = \left[B K_B^{-1}\cdot R_B - R_C\right] . \qquad (5.5)$$

Preconditioned Uzawa algorithms are defined as follows:

let $\Gamma = (<\varphi_i, \varphi_j>_{H^1})_{ij}$ *the Gram matrix of the finite element basis* $\{\varphi_1, \ldots, \varphi_n\}$ *and an incomplete Cholesky decomposition of* Γ : $\Gamma = P\cdot P^T - S$, *where S is often very sparse matrix (see J.A.Meijernik and H.A.van der Vorst, Math.Comp.1977).*

Formally, a preconditioned version of (5.3) with P as a preconditioner is given by:

$$\left[P^{-1}\left(B\cdot K_B^{-1} B^T - K_C\right)P^{-T}\right]\left[P^T\Delta p_k\right] = \left[P^{-1}\left(B K_B^{-1}\cdot R_B - R_C\right)\right] . \quad (5.6)$$

This is the Schur complement for the preconditioned version of (5.3) given by:

$$\begin{pmatrix} K_B & \overline{B^T} \\ \overline{B} & \overline{K_C} \end{pmatrix}\begin{pmatrix} \Delta U \\ \Delta \overline{p} \end{pmatrix} = \begin{pmatrix} R_B \\ R_C \end{pmatrix} \qquad (5.7)$$

where $\overline{B} = P^{-1}B$, $\overline{K_C} = P^{-T}K_C P^{-T}$, $\Delta\overline{p} = P^T\Delta p_k$.

Applying the Uzawa algorithm to the system (5.7) would produce a set of approximations $\Delta\overline{p}_k$. Setting $p_k = P^{-T}\cdot\Delta\overline{p}_k$ leads to an other preconditioned Uzawa algorithm:

for $k=0$ *until converge, do*

Solve $K_B\Delta U_{k+1} = R_B - \overline{B^T} + \delta_k$

Compute $\Delta\overline{p}_{k+1} = \Delta\overline{p}_k + \alpha P_k^{-1}\left(B\Delta U_{k-1} + K_C\Delta p_k - R_C\right)$

enddo.

Remarks. Good choices of the scalar α is determined from following observations:

equations (5.4)-(5.5) imply for the errors satisfy

$$\Delta p - \Delta p_k = \left(I - \alpha\left(B K_B^{-1} B^T - K_C\right)\right)^k (\Delta p - \Delta p_0)$$

must have $\rho\left(I - \alpha\left(BK_B^{-1} B^T - K_C\right)\right) < 1$ i.e. $0 < \alpha < \dfrac{2}{\lambda_1}$, where λ_1 is maximum

eigenvalue of $BK_B^{-1} B^T - K_C$.

We tried to overcome the difficulties due to the increasing of the condition number of the matrix $BK_B^{-1} B^T - K_C$, by applying the method of preconditioning presented.

REFERENCES

[1] **J. BRAMBLE, J. PASCIAK, and A. VASSILEV**, *Analysis of the inexact Uzawa algorithm for saddle point problems,* SIAM J., Numer. Anal. 34(1997), pp. 1072-1092

[2] **J. CÉA, R. GLOWINSKI,** *Methodes numeriques pour l'eculement laminare d'un fluid rigide viscoelastique incompressible*, Int. J. Comput. Math., Sect. B, 3, pp.225-255

[3] **X. CHEN,** *Global and superlinear convergence of inexact Uzawa methods for saddle point problems with nondifferentiable mappings*, SIAM J., Numer. Anal. 35(1998) No.3, pp 1130-1148

[4] **H. C. ELMAN and G.H. GOLUB,** *Inexact and preconditioned Uzawa algorithms for saddle point problems,* SIAM J., Numer. Anal. 31(1994), pp. 1645-1661

[5] **H. C. ELMAN**, *Relaxed and stabilised incomplete factorizations for non-self-adjoint linear systems,* BIT. 29(1989), pp. 890-915

[6] **N. POP,** *A dual analysis of a problem modelling a contact problem with friction,* Applied&Computing Mathematics, vol.II, (edit. L. Madarasz, V. Soltes), (1997), pp. 105-112

[7] **N. POP,** *A finite element solution for three-dimensional quasistatic frictional contact problem,* Revue Roumaine des Sciences Tehniques, Serie de Mécanique Applique, Academie Roumaine, 42, No. 1-2 (1997), pp. 209--219

[8] **P. WRIGERS, J. C. SIMO,** *A note an tangent stiffness for fully nonlinear contact problems,* Comm. In. App.Num. Meth., 1(1985), pp. 199-203

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