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Dedicated to Professor Ion PĂVAÎOÎIU on his 60th anniversary

COINCIDENCE POINTS FOR HYBRID CONTRACTIONS SATISFYING AN IMPLICIT RELATION

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Abstract. A general coincidence theorem for hybrid contractions satisfying an implicit relation is proved extending the main result from [1].

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1. Introduction. Let $(X,d)$ be a metric space. We denote by $CB(X)$ the set of all nonempty closed bounded subsets of $(X,d)$ and by $H$ the Hausdorff Pompeiu metric on $CB(X)$

$$H(A,B) = \max\{\sup_{x \in A} d(x,B); \sup_{x \in B} d(x,A)\}$$

where $A, B \in CB(X)$ and

$$d(x,A) = \inf_{y \in A} \{d(x,y)\}.$$

Let $A, B \in CB(X)$ and $k > 1$. In what follows the following well known fact will be used [3]: For each $a \in A$, there exists $b \in B$ such that $d(a,b) \leq kH(A,B)$.

Let $\delta(A,B) = \sup\{d(x,y) : x \in A \text{ and } y \in B\}$ for all $A, B \in CB(X)$. If $A$ consists of single valued "a" then we write $\delta(A,B) = \delta(a,B)$. If $\delta(A,B) = 0$ then $A = B = \{a\}$[5].

Let $S$ and $T$ be two self mappings of a metric space $(X,d)$. Sessa [6] defines $S$ and $T$ to be weakly commuting if $d(STx, TSx) < d(Tx, Sx)$ for all $x \in X$. Jungck [2] defines $S$ and $T$ to be compatible if $d(STx_n, TSx_n) \to 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim Tx_n = \lim Sx_n = x$ for some $x \in X$.

Clearly, commuting mapping are weakly commuting and weakly commuting mappings are compatible, but neither implication is reversible (Ex. 1[7] and ex. 2.2 [2]).

Let $(X,d)$ be a metric space, $f : X \to X$ and $S : X \to CB(X)$ single and multivalued mappings, respectively.
**Definition 1,** [1] The mappings \( f \) and \( S \) are said to be weakly commuting if for all \( x \in X, fSx \in CB(X) \) and \( H(Sfx, fSx) \leq d(fx, Sx) \).

**Definition 2,** [1] The mapping \( f \) and \( S \) are said to be compatible if
\[
\lim d(fy_n, Sfx_n) = 0 \quad \text{whenever} \quad \{x_n\} \quad \text{and} \quad \{y_n\} \quad \text{are sequences in} \quad X \quad \text{such that}
\]
\[
\lim fx_n - \lim y_n = z \quad \text{for some} \quad z \in X, \quad \text{where} \quad \{y_n\} \in Sx_n, \quad n=1,2,...
\]

Weakly commuting mappings \( f \) and \( S \) are compatible but implication is not reversible (Remark 1.1 and Ex. 1.1[1]).

**Theorem 1.** Let \((X,d)\) be a complete metric space and let \( S,T:(X,d) \rightarrow CB(X) \) be two multifunctions such that
\[
H^m(Sx, Ty) \leq c \frac{d^p(x, Sx) + d^p(y, Ty)}{\delta^m(x, Sx) + \delta^m(y, Ty)}
\]
holds for all \( x, y \) in \( X \) for which \( \delta^m(x, Sx) + \delta^m(y, Ty) \neq 0 \), where \( 0 < c < 1, \ m > 1, \ p \geq 2, \ m < p \). Then \( S \) and \( T \) have a common fixed point and \( F(S) = F(T) \), where \( F(S) = \{ x \in X : x \in Sx \} \).

In this paper we give a general coincidence theorem for hybrid contractions, i.e. contractive conditions involving single-valued and multi-valued mappings, satisfying an implicit relation which generalize Theorem 1.

**2. Implicit relations**

Let \( \mathcal{H}_5 \) be the set of all functions \( F(t_1, \ldots, t_5) : R^5_+ \rightarrow R \) with \( t_4 + t_5 \neq 0 \) satisfying the following conditions:

\( K_1 : F \) is decreasing in variables \( t_2, t_3 \) and non-decreasing in variables \( t_4 \) and \( t_5 \),

\( K_2 : \) there exists \( k > 1 \) and \( h \in (0,1) \) such that
\[
(K_a) : u \leq kt \quad \text{and} \quad F(t, u, v, u, v) \leq 0
\]
\[
(K_b) : u \leq kt \quad \text{and} \quad F(t, v, u, v, u) \leq 0
\]

implies \( u \leq hv \).

**Example 1.** \( F(t_1, \ldots, t_5) = t_1 - c \left[ \left( \frac{t_2^p + t_3^p}{t_4^m + t_5^m} \right)^{1/m} \right] \) where \( 0 < c < 1, \ m > 1, \ p \geq 2, \ m < p \) and \( t_4 + t_5 = 0 \).
$K_1$: Obviously.

$K_2$: Let $u > 0$, $u < kt$ and $F(t, u, v, u, v) \leq 0$ be, where $1 < k < (1/c)^{1/m}$. Then $u^p + u^mv^p - k^m cu^p - k^m cv^p \leq 0$. If $v = 0$, then $u = 0$, a contradiction. Thus $q^p(1 - k^m c) + q^m - k^m c < 0$ where $q = u/v$. Let $f : [0, \infty) \to \mathbb{R}$ be the function $f(q) = q^m(1 - k^m c) + q^m k^m c$. Then $f'(q) > 0$ for any $q > 0$, $f(0) < 0$ and $f(1) - 2(1 - k^m c) > 0$. Let $h \in (0, 1)$ be the root of the equation $f(q) = 0$, then $f(q) \leq 0$ for $q \leq h$, thus $u \leq hv$, where $h \in (0, 1)$.

Similarly, $u > 0$, $u < kt$ and $F(t, v, u, v, u) \leq 0$ implies $u < hv$.

If $u = 0$ then $u < hv$.

**Example 2.** $F(t_1, ..., t_5) = t_1^3 + t_2^2 - t_1 - \frac{(bt_2 + ct_3)^2}{t_4 + t_5}$ where $0 < b + c < 1$ and $t_4 + t_5 \neq 0$.

$K_1$: Obviously.

$K_2$: Let $u > 0$, $u < kt$ and $F(t, u, v, u, v) \leq 0$ where $1 < k \min \left\{ \frac{1}{b^2}, \frac{1}{c^2}, 2/(b + c)^2 \right\}$.

Then $t^3 + t^2 + t - \frac{(bu + cv)^2}{u + v} \leq 0$ which implies $t - \frac{(bu + cv)^2}{u + v} \leq 0$.

Then $u < kt \leq k \frac{(bu + cv)^2}{u + v}$ implies $u^2(1 - kb^2) + uv(1 - 2bck) - c^2v^2k \leq 0$.

If $v = 0$ then $u = 0$, a contradiction. Thus $q^2(1 - kb^2) + q(1 - 2bck) - k^2c^2 < 0$ where $q = u/v$. Let $f : [0, \infty) \to \mathbb{R}$ be the function $f(q) = (1 - kb^2) + uv(1 - 2bck)q - c^2k$.

Then $f(0) < 0$ and $f(1) - 2 - k(b + c)^2 > 0$. Let $h_1 \in (0, 1)$ the root of the equation $f(t) = 0$, then $f(t) \leq 0$ for $t \leq h_1$ and thus $u \leq h_1$. Similarly, $u > 0$, $u < kt$ and $F(t, v, u, v, u) \leq 0$ implies $u \leq h_2v$, where $h_2 \in (0, 1)$. Then $u < hv$, where $h = \max\{h_1, h_2\}$ and $h \in (0, 1)$. If $u = 0$ then $u < hv$. 
3. Main result

**Theorem 2.** Let \((X,d)\) be a complete metric space. Let \(f,g:X \to X\) be continuous single-valued mappings and \(S,T:X \to \text{CB}(X)\) be \(H\)-continuous multi-valued mappings such that

1. \(T(X) \subset f(X)\) and \(S(X) \subset g(X)\),
2. The pairs \(\{f, S\}\) and \(\{g, T\}\) are compatible,
3. \(F(H(Sx,Ty), d(fx,Sx), d(y,Ty), \delta(fx,Sx),\delta(gy,Ty)) < 0\) for all \(x,y \in X\) with \(\delta(fx,Sx) + \delta(gy,Ty) > 0\) for \(F \in K_3\), then

1°. \(f\) and \(S\) have a coincidence point and \(g\) and \(T\) have a coincidence point or
2°. \(f\) and \(S\) and \(g\) and \(T\) have a common coincidence point.

**Proof.** Let \(x_0\) be an arbitrary but fixed element of \(X\). Since \(T(X) \subset f(X)\) and \(k > 1\) there exist \(x_1 \in X\) such that \(y_1 = gx_1 \in Sx_0\). Since \(T(X) \subset f(X)\) and \(k > 1\) there exist \(y_2 = fx_2 \in Tx_1\) such that

\[
d(y_1, y_2) = d(gx_1, fx_2) < kH(Sx_0, Tx_1).
\]

Similarly, there exists a point \(x_3 \in X\) such that \(y_3 = gx_3 \in Sx_2\) and

\[
d(y_2, y_3) = d(fx_2, gx_3) < kH(Sx_2, Tx_1).
\]

Inductively, we can obtain the sequences \(\{x_n\}, \{y_n\}\) such that

1. \(y_{2n+1} = gx_{2n+1} \in Sx_{2n}\),
2. \(y_{2n-2} = fx_{2n-2} \in Tx_{2n-1}\),
3. \(d(y_{2n+1}, y_{2n}) < kH(Sx_{2n}, Tx_{2n-1})\) and
4. \(d(y_{2n+1}, y_{2n+2}) < kH(Sx_{2n}, Tx_{2n+1})\), for every \(n \in \mathbb{N}\).

First suppose that some \(n \in \mathbb{N}\), \(\delta(fx_n, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0\).

Then \(fx_{2n} \in Sx_{2n}\) and \(gx_{2n+1} \in Tx_{2n+1}\) and so \(x_{2n}\) is a coincidence point of \(f\) and
S and \( x_{2n+1} \) is a coincidence point of \( g \) and \( T \).

Similarly, \( \delta(fx_{2n-2}, Sx_{2n+1}) + \delta(x_{2n-1}, Tx_{2n-1}) = 0 \) for some \( n \in \mathbb{N} \), implies that \( x_{2n-2} \) is a coincidence point of \( f \) and \( S \) and \( x_{2n-1} \) is a coincidence point of \( g \) and \( T \). Now, suppose that \( \delta(fx_{2n+1}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) \neq 0 \) for \( n \in \mathbb{N} \).

Then by (3.3) we have successively

\[
F(H(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n-1}), \delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1})) \leq 0
\]

(5) \( F(H(Sx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n-2}), d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n-2})) \leq 0 \).

If \( d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n-2}) = 0 \) then \( fx_{2n} = gx_{2n+1} \in Sx_{2n} \) and

\( gx_{2n+1} = fx_{2n+2} \in Tx_{2n+1} \) and thus \( x_{2n} \) is a coincidence point of \( f \) and \( S \) and \( x_{2n+1} \) is a coincidence point of \( g \) and \( T \). Let \( d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n-2}) \neq 0 \) for \( n \in \mathbb{N} \). Then by condition \( (K_\delta) \), (4) and (5) we have

\[
(6) \quad d(y_{2n-1}, y_{2n+2}) \leq h d(y_{2n}, y_{2n-1}).
\]

Similarly, by (3.3) we have

(7) \( F(H(Tx_{2n+1}, Sx_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2})) \leq 0 \)

If \( d(y_{2n+2}, y_{2n+3}) + d(y_{2n+1}, y_{2n+2}) = 0 \), then \( gx_{2n+1} = fx_{2n+2} \in Tx_{2n+1} \) and

\( fx_{2n+2} = gx_{2n+3} \in Sx_{2n+2} \) and thus \( x_{2n+1} \) is a coincidence point \( f \) and \( S \) and \( x_{2n+2} \) is a coincidence point of \( g \) and \( T \). Let \( d(y_{2n+2}, y_{2n+3}) + d(y_{2n+1}, y_{2n+2}) \neq 0 \) for \( n \in \mathbb{N} \).

Then by conditions \( (K_\delta) \), (7) we have

\[
(8) \quad d(y_{2n+2}, y_{2n+3}) \leq h d(y_{2n+1}, y_{2n+2}).
\]

By (6) and (8) it follows that the sequence \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \((X, d)\) is a complete metric space, let \( \lim gx_{2n+1} = \lim fx_{2n} = z \). Now, we will prove that \( fz \in Sz \), that is, \( z \) is a coincidence point of \( f \) and \( S \). For every \( n \in \mathbb{N} \), we have
(9) \[ d(fgx_{2n+1},Sz) \leq d(fgx_{2n+1},Sfx_{2n}) + H(Sfx_{2n},Sz). \]

It follows from the H-continuity of S that

(10) \[ \lim H(Sfx_{2n},Sz) = 0 \]

since \( fx_{2n} \to z \) as \( n \to \infty \). Since \( f \) and S are compatible mapping and

\[ \lim f(u_n) = \lim v_n = z \] where \( v_n = gx_{2n+1} \in Sx_{2n} \) and \( u_n = x_{2n} \), we have

(11) \[ \lim d(fv_n,Sfx_{2n}) = \lim d(fgx_{2n+1},Sfx_{2n}) = 0. \]

Thus from (9),(10) and (11) we have \( \lim d(fgx_{2n+1},Sz) = 0 \) and so, from

\[ d(fz,Sz) \leq d(fz,fgx_{2n+1}) + (fgx_{2n+1},Sz) \] and the continuity, it follows that

\[ d(fz,Sz) = 0, \] which implies that \( fz \in Sz \) since \( Sz \) is a closed subset of \( X \).

Similarly, we can prove that \( gz \in Tz \), that is, \( z \) is a coincidence point of \( g \) and \( T \).

This completes the proof.

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