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Dedicated to Professor Ion PĂVĂLOIU on his 60^{th} anniversary

COINCIDENCE POINTS FOR HYBRID CONTRACTIONS SATISFYING AN IMPLICIT RELATION

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Abstract. A general coincidence theorem for hybrid contractions satisfying an implicit relation is proved extending the main result from [1].

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1. Introduction. Let (X,d) be a metric space. We denote by CB(X) the set of all nonempty closed bounded subsets of (X,d) and by H the Hausdorff Pompeiu metric on CB(X)

$$H(A,B) = \max\{\sup_{x\in A} d(x,B);\ \sup_{x\in B} d(x,A)\}$$

where $A,B\in CB(X)$ and

$$d(x,A) = \inf_{y\in A}\{d(x,y)\}.$$

Let $A,B\in CB(X)$ and $k>1$. In what follows the following well known fact will be used [3]: For each $a\in A$, there exists $b\in B$ such that $d(a,b)\leq kH(A,B)$.

Let $\delta(A,B)=\sup\{d(x,y): x\in A \text{ and } y\in B\}$ for all $A,B\in CB(X)$. If A consists of single valued "a" then we write $\delta(A,B)=\delta(a,B)$. If $\delta(A,B)=0$ then $A=B=\{a\}$ [5]. Let S and T be two self mappings of a metric space (X,d). Sessa [6] defines S and T to be weakly commuting if $d(STx, TSx)\leq d(Tx, Sx)$ for all $x\in X$. Jungck [2] defines S and T to be compatible if $d(STx_n, TSx_n)\to 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Tx_n = \lim Sx_n = x$ for some $x\in X$. Clearly, commuting mapping are weakly commuting and weakly commuting mappings are compatible, but neither implication is reversible (Ex.1[7] and ex.2.2 [2]).

Let (X,d) be a metric space, $f: X\to X$ and $S: X\to CB(X)$ single and multivalued mappings, respectively.

Definition 1,[1]. The mappings f and S are said to be weakly commuting if for all $x \in X, fSx \in CB(X)$ and $H(Sfx, fSx) \le d(fx, Sx)$.

Definition 2, [1] The mapping f and S are said to be compatible if $\lim d(fy_n, Sfx_n) = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$\lim fx_n = \lim y_n = z$ for some $z \in X$, where $\{y_n\} \in Sx_n$, $n = 1,2...$.

Weakly commuting mappings f and S are compatible but implication is not reversible (Remark 1.1 and Ex.1.1[1])

Theorem 1. *Let (X,d) be a complete metric space and let $S,T:(X,d) \to CB(X)$ be two multifunctions such that*

$$H^m(Sx, Ty) \le c \frac{d^p(x,Sx) + d^p(y,Ty)}{\delta^{p-m}(x,Sx) + \delta^{p-m}(y,Ty)}$$

holds for all x,y in X for which $\delta^{p-m}(x,Sx) + \delta^{p-m}(y,Ty) \neq 0$, where $0<c<1$, $m>1$, $p \ge 2$, $m<p$. Then S and T have common fixed point and $F(S)=F(T)$, where

$$F(S) = \{x \in X : x \in Sx\}.$$

In this paper we give a general coincidence theorem for hybrid contractions, i.e. contractive conditions involving single-valued and multi-valued mappings, satisfying an implicit relation which generalize Theorem 1.

2. Implicit relations

Let $\mathscr{K}_5$ be the set of all functions $F(t_1, \ldots, t_5) : R_+^5 \to R$ with $t_4 + t_5 \neq 0$ satisfying the following conditions:

K_1: F is decreasing in variables t_2, t_3 and non-decreasing in variables t_4 and t_5 ,

K_2: there exists $k>1$ and $h \in (0,1)$ such that

$$(K_a) : u \le kt \text{ and } F(t,u,v,u,v) \le 0$$

$$(K_b) : u \le kt \text{ and } F(t,v,u,v,u) \le 0$$

implies $u \le hv$.

Example 1. $F(t_1,\ldots,t_5) = t_1 - c\left[(t_2^p + t_3^p)/(t_4^{p-m} + t_5^{p-m})\right]^{1/m}$ where $0<c<1$, $m>1$, $p \ge 2$, $m<p$ and $t_4 + t_5 \neq 0$.

K_1: Obviously.

K_2: Let $u>0, u\le kt$ and $F(t,u,v,u,v)\le 0$ be, where $1<k<(1/c)^{1/m}$. Then $u^p+u^m v^{p-m}-k^m c u^p-k^m c v^p\le 0$. If $v=0$, then $u=0$, a contradiction. Thus $q^p(1-k^m c)+q^m-k^m c\le 0$ where $q=u/v$. Let $f:[0,\infty)\to R$ be the function $f(q)=q^m(1-k^m c)+q^m-k^m c$. Then $f'(q)>0$ for any $q>0$, $f(0)<0$ and $f(1)=2(1-k^m c)>0$. Let $h\in(0,1)$ be the root of the equation $f(q)=0$, then $f(q)\le 0$ for $q\le h$, thus $u\le hv$, where $h\subset(0,1)$.

Similarly, $u>0, u\le kt$ and $F(t,v,u,v,u)\le 0$ implies $u\le hv$.
If $u=0$ then $u\le hv$.

Example 2. $F(t_1,\dots,t_5)=t_1^3+t_1^2+t_1-\dfrac{(bt_2+ct_3)^2}{t_4+t_5}$ where $0<b+c<1$ and $t_4+t_5\neq 0$.

K_1: Obviously. K_2: Let $u>0, u\le kt$ and $F(t,u,v,u,v)\le 0$ where $1<k\ min\{1/b^2,\ 1/c^2, 2/(b+c)^2\}$.

Then $t^3+t^2+t-\dfrac{(bu+cv)^2}{u+v}\le 0$ which implies $t-\dfrac{(bu+cv)^2}{u+v}\le 0$.

Then $u\le kt\le k\dfrac{(bu+cv)^2}{u+v}$ implies $u^2(1-kb^2)+uv(1-2bck)-c^2v^2k\le 0$.

If $v=0$ then $u=0$, a contradiction. Thus $q^2(1-kb^2)+q(1-2bck)-kc^2\le 0$ where $q=u/v$. Let $f:[0,\infty)\to R$ be the function $f(q)=(1-kb^2)+uv(1-2bck)q-c^2k$. Then $f(0)<0$ and $f(1)=2-k(b+c)^2>0$. Let $h_1\in(0,1)$ the root of the equation $f(t)=0$, then $f(t)\le 0$ for $t\le h_1$ and thus $u\le h_1$. Similarly, $u>0, u\le kt$ and $F(t,v,u,v,u)\le 0$ implies $u\le h_2v$, where $h_2\in(0,1)$. Then $u\le hv$, where $h=max\{h_1,h_2\}$ and $h\subset(0,1)$. If $u=0$ then $u\le hv$.

3. Main result

Theorem 2. *Let (X,d) be a complete metric space. Let* $f, g : X \to X$ *be continuous single-valued mappings and* $S, T : X \to CB(X)$ *be H-continuous multi-valued mappings such that*

(3.1) $T(X) \subset f(X)$ *and* $S(X) \subset g(X)$,

(3.2) The pairs {f, S} and {g,T} are compatible,

(3.3) $F(H(Sx, Ty), d(fx, Sx), d(y, Ty), \delta(fx, Sx), \delta(gy, Ty)) \leq 0$ *for all* $x, y \in X$ *with* $\delta(fx, Sx) + \delta(gy, Ty) \neq 0$ *for* $F \in K_5$, *then*

1^0. f and S have a coincidence point and g and T have a coincidence point or

2^0. f and S and g and T have a common coincidence point.

Proof. Let x_0 be an arbitrary but fixed element of X. Since $T(X) \subset f(X)$ and $k>1$ there exist $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0$. Since $T(X) \subset f(X)$ and k>1 there exist $y_2 = fx_2 \in Tx_1$ such that

$$d(y_1, y_2) = d(gx_1, fx_2) \leq k\, H(Sx_0, Tx_1).$$

Similarly, there exists a point $x_3 \in X$ such that $y_3 = gx_3 \in Sx_2$ and

$$d(y_2, y_3) = d(fx_2, gx_3) \leq k\, H(Sx_2, Tx_1).$$

Inductively, we can obtain the sequences $\{x_n\}$, $\{y_n\}$ such that

(1) $y_{2n+1} = gx_{2n+1} \in Sx_{2n}$,

(2) $y_{2n+2} = fx_{2n+2} \in Tx_{2n+1}$,

(3) $d(y_{2n+1}, y_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1})$ and

(4) $d(y_{2n+1}, y_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1})$, for every $n \in \mathbb{N}$.

First suppose that some $n \in \mathbb{N}$, $\delta(fx_n, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0$.

Then $fx_{2n} \in Sx_{2n}$ and $gx_{2n+1} \in Tx_{2n-1}$ and so x_{2n} is a coincidence point of f and

S and x_{2n+1} is a coincidence point of g and T.

Similarly, $\delta(fx_{2n+2}, Sx_{2n+2}) + \delta(x_{2n+1}, Tx_{2n+1}) = 0$ for some $n \in \mathbb{N}$, implies that x_{2n+2} is a coincidence point of f and S and x_{2n-1} is a coincidence point of g and T. Now, suppose that $\delta(fx_{2n+1}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) \neq 0$ for $n \in \mathbb{N}$. Then by (3.3) we have succesively

$$F(H(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n+1}), \delta(fx_{2n}, Sx_{2n}), \delta(gx_{2n+1}, Tx_{2n+1})) \leq 0$$

(5) $F(H(Sx_{2n}, Tx_{2n+1}), d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n+2}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n+2})) \leq 0$.

If $d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n}) = 0$ then $fx_{2n} = gx_{2n-1} \in Sx_{2n}$ and $gx_{2n+1} = fx_{2n+2} Tx_{2n+1}$ and thus x_{2n} is a coincidence point of f and S and x_{2n+1} is a coincidence point of g and T. Let $d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n+2}) \neq 0$ for $n \in \mathbb{N}$. Then by condition (K_h), (4) and (5) we have

(6) $$d(y_{2n+1}, y_{2n+2}) \leq h\, d(y_{2n}, y_{2n+1}) .$$

Similarly, by (3.3) we have

(7) $F(H(Tx_{2n+1}, Sx_{2n+2}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n-3}), d(y_{2n+1}, y_{2n+2})) \leq 0$

If $d(y_{2n+2}, y_{2n+3}) + d(y_{2n+1}, y_{2n+2}) = 0$, then $gx_{2n-1} = fx_{2n+2} \subset Tx_{2n+1}$ and $fx_{2n+2} = gx_{2n+3} \in Sx_{2n+2}$ and thus x_{2n+1} is a coincidence point f and S and x_{2n+2} is a coincidence point of g and T. Let $d(y_{2n+2}, y_{2n+3} + d(y_{2n+1}, y_{2n+2}) \neq 0$ for $n \in \mathbb{N}$. Then by conditions (K_a), (7) we have

(8) $$d(y_{2n+2}, y_{2n+3}) \leq h\, d(y_{2n-1}, y_{2n-2}).$$

By (6) and (8) it folows that the sequence $\{y_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete metric space, let $lim\, gx_{2n+1} = lim\, fx_{2n} = z$. Now, we will prove that $fz \in Sz$, that is, z is a coincidence point of f and S. For every $n \in \mathbb{N}$, we have

(9) $$d(fgx_{2n+1}, Sz) \leq d(fgx_{2n+1}, Sfx_{2n}) + H(Sfx_{2n}, Sz).$$

It in follows from the H-continuity of S that

(10) $$lim H(Sfx_{2n}, Sz) = 0$$

since $fx_{2n} \to z$ as $n \to \infty$. Since f and S are compatible mapping and $lim f(u_n) = lim v_n = z$ where $v_n = gx_{2n+1} \in Sx_{2n}$ and $u_n = x_{2n}$, we have

(11) $$lim\, d(fv_n, Sfx_{2n}) = lim\, d(fgx_{2n+1}, Sfx_{2n}) = 0.$$

Thus from (9),(10) and (11) we have $lim\, d(fgx_{2n+1}, Sz) = 0$ and so, from $d(fz, Sz) \leq d(fz, fgx_{2n+1}) + (fgx_{2n+1}, Sz)$ and the continuity, it follows that $d(fz, Sz) = 0$, which implies that $fz \in Sz$ since Sz is a closed subset of X. Similarly, we can prove that $gz \in Tz$, that is, z is a coincidence point of g and T. This completes the proof.

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