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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

THE RELATIVIST EXPRESSION OF THE ACCELERATION PRODUCED BY A CENTRAL SPHERICAL SYMMETRIC BODY ON MATERIAL POINT ROTATING AROUND THAT

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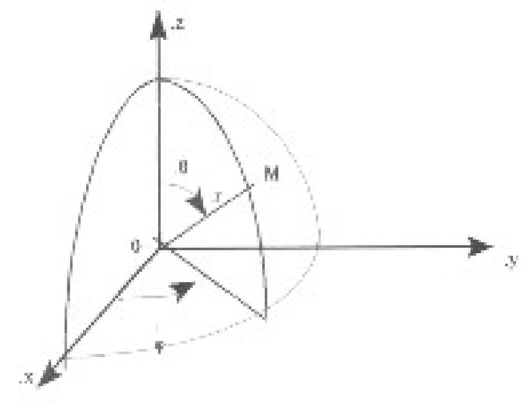
Abstract. In this paper is presented the introduction of the centrifugal acceleration into the relativist expression of the acceleration produced by a central body with spherical symmetry on a material point that rotates around Oz axis.

1. INTRODUCTION

If we consider a spherical symmetric central body having its center in the origin of a Cartesian frame Oxyz. One material point M, situated in the neighbourhood of a central body has its spherical coordinates generalized in the quadridimensional space:

$$x^0 = t \; ; \; x^1 = r \; ; \; x^2 = \theta \; ; \; x^3 = \varphi \tag{1}$$

where x^0 is the time coordinate and x^1, x^2, x^3 are the space coordinates.



For calculations we use geometrized units physical sizes (with index ph), that can be obtained from these by multiplying them with an universal constants combination, using:

c = velocity of lihght; G = universal constant of gravitation

time $t = c\, t_{ph}\, (m)$,

angular velocity $\omega = \dfrac{\omega_{ph}}{c}\, (m^{-1})$, (2)

mass $m = \dfrac{GM_{ph}}{c^2}\, (m)$,

acceleration $a = \dfrac{a_{ph}}{c^2}\, (m^{-1})$.

When we have a spherical symmetry, the space-time metrics has the form:

$$ds^2 = g_{00}\,(dx^0)^2 + g_{11}\,(dx^1)^2 + g_{22}\,(dx^2)^2 + g_{33}\,(dx^3)^2 \ , \tag{3}$$

when the fundamental metric tensor has the covariant components

$$g_{ij} = g_{ij}\,(x^1, x^2) \ , \tag{4}$$

and the covariant ones:

$$g^{ij} = \frac{1}{g_{ij}} \ , \tag{5}$$

where we note with

$$u^k = \frac{dx^k}{ds} \ , \tag{6}$$

the quadrivelocity components.

Taking into consideration that the material point M has a rotation around Oz axis

$$\frac{dx^3}{dx^0} = \frac{d\varphi}{dt} = \omega \ , \tag{7}$$

$$u\,(u^0, 0, 0, u^3) = u\,(u^0, 0, 0, \omega u^0) \ , \tag{8}$$

with

$$u^0 = (g_{00} + \omega^2 g_{33})^{-1/2} \quad . \tag{9}$$

Keeping the space coordinates constants, we get:

$$ds = d\tau \quad , \tag{10}$$

where τ represents the time measured inside the point M.

2. FORMULATION OF THE PROBLEM AND THE EXPRESSION OF THE ACCELERATION

The equation of geodesics

$$\frac{d^2 x^i}{ds^2} = -\Gamma^i_{km} \frac{dx^k}{ds} \frac{dx^m}{ds} \quad , \tag{11}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{il} (g_{lk,m} + g_{lm,k} - g_{km,l}) \tag{12}$$

represents the Christoffel's symbols of the second kind, the notations are those used for the tensorial calculus.

At the repetition of the index, the sum is made after that index, and the comma followed by an index l, shows us that the derivation is done by the x^l coordinate. We obtain:

$$a^\alpha = \frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{km} u^k u^m \; , \quad \alpha \in \{1,2,3\} \; , \tag{13}$$

that means

$$a^\alpha = -(u^0)^2 \left(\Gamma^i_{00} + \omega^2 \Gamma^i_{33} \right). \tag{14}$$

In our situation we have the following non-null components involved in the formula (14) for the Christoffel's symbols:

$$\Gamma^1_{00} = -\frac{g_{00,1}}{2 g_{11}} \; ; \quad \Gamma^2_{00} = -\frac{g_{00,2}}{2 g_{22}}$$
$$\Gamma^1_{33} = -\frac{g_{33,1}}{2 g_{11}} \; ; \quad \Gamma^2_{33} = -\frac{g_{33,2}}{2 g_{22}} \tag{15}$$

We obtain

$$a^1 = a^r = \frac{g_{00,1} + \omega^2 g_{33,1}}{2g_{11}(g_{00} + \omega^2 g_{33})},$$

$$a^2 = a^\theta = \frac{g_{00,2} + \omega^2 g_{33,2}}{2g_{22}(g_{00} + \omega^2 g_{33})}, \tag{16}$$

$$a^3 = a^\varphi = 0 \ .$$

The quadrate of the acceleration's module is equal to

$$|a|^2 = \gamma_{\alpha\beta}\, a^\alpha a^\beta \ , \tag{17}$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}} \ , \tag{18}$$

are the components of the tridimensional space metric tensor

$$\gamma_{11} = -g_{11} \ , \ \gamma_{22} = -g_{22} \ , \ \gamma_{12} = 0 \ , \tag{19}$$

and

$$|\bar{a}|^2 = -g_{11}(a^1)^2 - g_{22}(a^2)^2 = -\frac{1}{g_{11}}\left[\frac{g_{00,1} + \omega^2 g_{33,1}}{2(g_{00} + \omega^2 g_{33})}\right]^2 - \frac{1}{g_{22}}\left[\frac{g_{00,2} + \omega^2 g_{33,2}}{2(g_{00} + \omega^2 g_{33})}\right]^2 . \tag{20}$$

The acceleration vector in point M is equal to

$$\vec{a} = -\frac{g_{00,1} + \omega^2 g_{33,1}}{2\sqrt{-g_{11}}\,(g_{00} + \omega^2 g_{33})}\,\vec{\rho} - \frac{g_{00,2} + \omega^2 g_{33,2}}{2\sqrt{-g_{22}}\,(g_{00} + \omega^2 g_{33})}\,\vec{\tau} \ , \tag{21}$$

where $\vec{\rho} = \frac{\vec{r}}{r}$ is the radial and $\vec{\tau}$ is the meridian tangent versor one.

If the central body has not electrical charge, then in its neigbourhood is applied the Schwarzschild metrics:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \ . \qquad (22)$$

From (22) and (21) results

$$\vec{a} = \frac{\sqrt{1 - \frac{2m}{r}}\left(-\frac{m}{r^2} + \omega^2 r \sin^2\theta\right)\bar{\rho} + \omega^2 r \sin\theta \cos\theta \, \bar{\tau}}{1 - \frac{2m}{r} - \omega^2 r^2 \sin^2\theta} \ . \qquad (23)$$

By passing to the physical units we have

$$\vec{a} = \frac{\sqrt{1 - \frac{2GM}{c^2 r}}\left(-\frac{GM}{r^2} + \omega^2 r \sin^2\theta\right)\bar{\rho} + \omega^2 r \sin\theta \cos\theta \, \bar{\tau}}{1 - \frac{2GM}{c^2 r} - \frac{\omega^2 r^2}{c^2}\sin^2\theta} \ . \qquad (24)$$

In normal conditions

$$\frac{2GM}{c^2 r} \approx 0 \quad \text{and} \quad \frac{\omega^2 r^2}{c^2} \approx 0 \ , \qquad (25)$$

we obtain

$$\vec{a} = -\frac{GM}{r^2}\bar{\rho} + \omega^2 r \sin\theta \,(\sin\theta \, \bar{\rho} + \cos\theta \, \bar{\tau}) = \vec{g} + \vec{a}_{cf} \ , \qquad (26)$$

where $\vec{g}$ is the gravitation acceleration and $\vec{a}_{cf}$ is the centrifugal one.

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