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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

THE MODELLING OF INCOMPRESSIBLE FLUID MOVEMENT USING HELMHOLTZ'S DIAGRAM

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1. INTRODUCTION.

We consider the steady flow of incompressible fluid, which moves in the circular domain $r^2 \le x^2 + y^2 \le R^2$ asishown (Figure1)

We assume that in a very short time, the interior circle travels almost tangential to the exterior circle. If the circles were tangent, the velocity of thein the contact point is zero [1].

We study the flow immediately after the two circles move away(see.Figure 2). Futher we make the following hypotheses:

1. The fluid attacs the interior solid circle with velocity V_∞ in a point O, (see. Figure 3). The stream lines, DO, branches out in two streames lines along the solid $OP_1 \equiv L_1$ and $OP_2 \equiv L_2$.

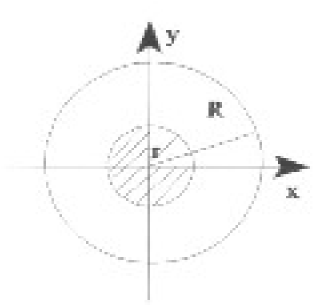

Figure 1

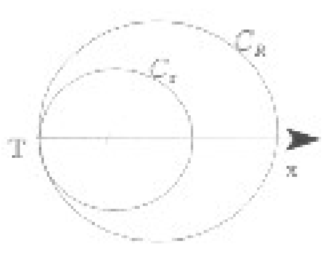

Figure 2

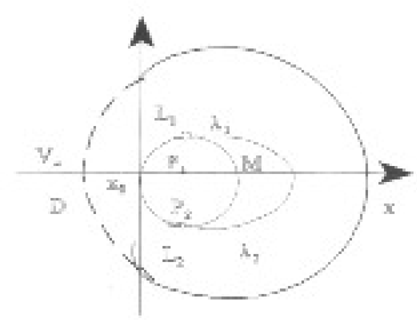

Figure 3

2. The points P_1 and P_2 are separates points of the interior cylinder's jet.

3. From points P_1 and P_2 the fluid particles move on the stream lines λ_1 and λ_2.

4. The area M included between λ_1 and λ_2 stream is called, in Helmholtz's diagram, dead area, where the fluid is in rest (reported to the Cr circle) [3] .

The λ_1 and λ_2 lines are discontinuity lines for the tangential speed. The velocity's modulus on λ_1 and λ_2 is equal to V_∞ .

Remark: in Helmholtz's diagram, λ_1 and λ_2 stream lines extend to infinity, hypothesis which we consider to be acceptable, the interest area of the flow in this case is ($OP_1\ \lambda_1$) and
($OP_2\ \lambda_2$).

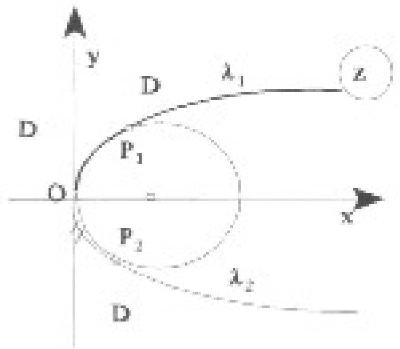

Figure 4

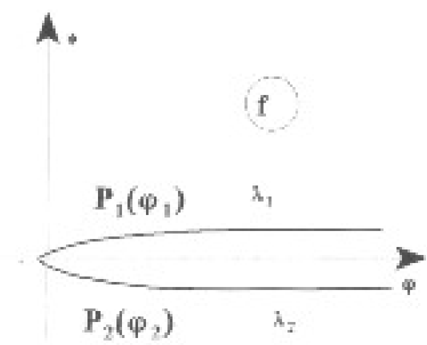

Figure 5

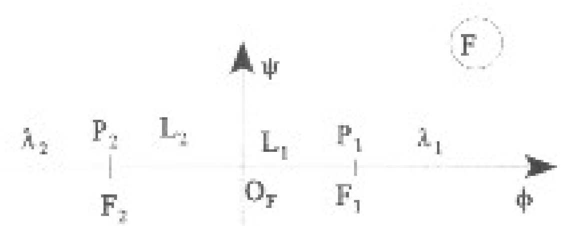

Figure 6

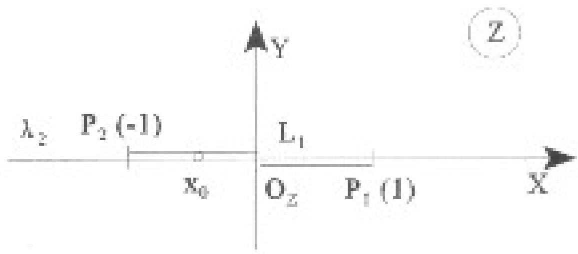

Figure 7

May $z=x+iy$ be and the two-dimension flow with its origin in the critic point O.

In the area D the movement is considered potential, with the complex potential $f(z) = \varphi + i\psi$ (see. Figure 4)

To study this we will use the method of T. Levi-Civita.

The method starts with the determination of the conforma mapping of the movement's domain from the plan f(z) on the interior of a semicircle with the unity radius from the plan ζ , in order for the solid walls L_1 and L_2 to be represented on the semicircle, and for the free stream lines λ_1 and λ_2 to be represented on the diameter of the semicircle.

To this purpose there we shall use the following transformations (see. Figure 5 to 7):

$$f=F^2,(1) \quad F=\phi+i\psi=\pm\sqrt{f}=\begin{cases} f=0=F,\ f=\infty=F \\ P_1\to F_1=\sqrt{\varphi_1},\ P_2\to F_2=-\sqrt{\varphi_2} \end{cases}$$

$$F=aZ+b=a(Z+\cos\sigma_0)\ ,\quad F_1\to X_1=+1,\ F_2\to X_2=-1$$

The transformation $F=a(Z+\cos\sigma_0)$ moves the F_1F_2 segment from the plan $Z=X+iY$ on the segment $[-1,1]$ of the real axis X, and the origin $F=0$ into the point $X_0=-\cos\sigma_0$ (a and σ_0 are real, unknown values).

We will use the transformation of Jucovscki

(2) $Z=-\frac{1}{2}(\zeta+\frac{1}{\xi})$ or $\zeta=-Z+\sqrt{Z^2-1}$, $(\zeta=\xi+i\eta)$ which transforms the $[-1,1]$ segment of the real axis X in on the semicircle from the planζ with radius 1

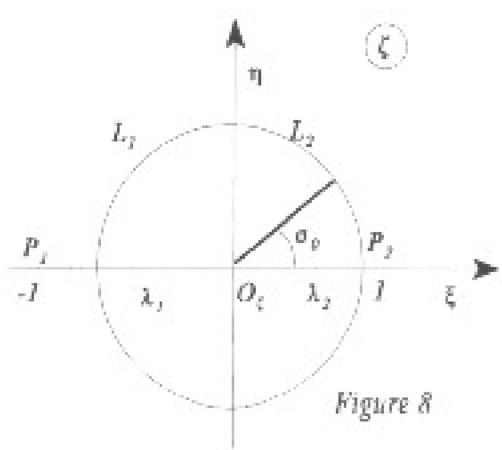

Figure 8

(for $\zeta=e^{i\sigma_0}$ we have

$Z=-\cos\sigma_0$, $\cos\sigma_0=1$ for $\sigma_0=0$.

If ζ describes the segment (-1, 0) we have

$$\zeta=\xi+i\eta \quad Z=-\frac{1}{2}(\xi+i\frac{1}{\xi}) .$$

The passing function from plan ζ to plan f will be $f=F^2=a^2(Z+\cos\sigma_0)^2$ (fig.7) or $f=a^2\left[\cos\sigma_0-\frac{1}{2}(\zeta+\frac{1}{\zeta})\right]^2$.

We have to be determined the a and σ_0 constants.

Futher, we use the known notation $\omega=\theta+i\tau$

where

$$(4)\qquad \begin{cases} (a)\ \bar{v}=\dfrac{df}{dz}=V_\infty e^{-i\omega}=V_\infty e^{\tau}\cdot e^{-i\theta}; \\ (b)\ \omega_{\zeta=0}=\omega_{\xi=\infty}=0; \\ (c)\ \omega_{\zeta=e^{i\sigma_0}}=\omega_{z=0}=\infty; \\ (d)\ \text{on the free lines } |V|=V_\infty \Rightarrow \tau=0 \text{ şi } Im\,\omega=0,\ -1<\zeta<1 \end{cases}$$

(e) Using the Schwarz principle, we extend the function $\omega(\zeta)$, wich is olomorphe in the entire unit circle.

Searching $\omega(\zeta)$ as a series z we get

$$(5)\qquad \omega(\zeta)=\frac{1}{\pi}\int_0^{2\pi}\theta(\sigma)\cdot\frac{\zeta e^{-i\sigma}}{1-\zeta e^{-i\sigma}}d\zeta\ .$$

2. THE DETERMINATION OF THE CONFORMABLE REPRESENTING FUNCTION IN THE CASE OF THE GIVEN PHYSIC MODEL

We consider the equation

$$(6)\qquad dz=\frac{1}{V_\infty}e^{i\omega}\cdot df=-\frac{a^2}{V_\infty}e^{i\omega}\left[\cos\sigma_0-\frac{1}{2}(\zeta+\frac{1}{\zeta})\right](1-\frac{1}{\zeta^2})d\zeta$$

where $\cos\sigma_0=\cos 0=1.$. After integration we obtain

$$(7)\qquad z=\frac{a^2}{2V_\infty}\int_{\zeta_0}^{\zeta}e^{i\omega(\zeta)}(\zeta+\frac{1}{\zeta}-2)(\zeta-\frac{1}{\zeta})\frac{d\zeta}{\zeta}$$

Between the points $z=x+iy$ from L_1+L_2 and the homologous $\zeta=e^{i\sigma}$ from the circle with unit radius we have:

$$(8)\qquad \begin{cases} x = \dfrac{-2a^2}{V_\infty}\displaystyle\int_{\sigma_0}^{\sigma} e^{\tau(\sigma)}(\cos\sigma - 1)\sin\sigma\cos\theta(\sigma)d\sigma \\ y = -\dfrac{2a^2}{V_\infty}\displaystyle\int_{\sigma_0}^{\sigma} e^{-\tau(\sigma)}\cdot(\cos\sigma - 1)\sin\sigma\sin\theta(\sigma)d\sigma \end{cases}$$

The detachment points P_1 and P_2 of the fluid particles are obtained respectively for $\sigma = \pi, \sigma = 0$ in the relations (8). In the present problem's case, using also the symmetry principle of Schwarz, we presume, for example that:

$$(9)\qquad \theta = \begin{cases} \dfrac{\pi}{4}\ for\ L_1;\ \dfrac{\pi}{2} < \sigma < \dfrac{3\pi}{2} \\ -\dfrac{\pi}{4}\ for\ L_2;\ -\dfrac{\pi}{2} < \sigma < \dfrac{\pi}{2} \end{cases}$$

Now, using (5) there have

$$(10)\qquad \omega(\zeta) = -\frac{i}{2}\ln\frac{1-i\zeta}{1+i\zeta}$$

and the complex potential for $\sigma_0 = 0$ will be

$$(11)\qquad f(\zeta) = a^2[1 - \frac{1}{2}(\zeta + \frac{1}{\zeta})]^2 .$$

3. THE DETERMINATION OF $e^{-\tau}$ AND A^2 CONSTANTS

Knowing in the physical plan the detachment points P_1 and P_2 for example: $P_1(\frac{r}{2}, r\frac{\sqrt{2}}{2})$ for $\sigma_0 = 0$ and using relation (8) we will obtain for the determination of constant a^2:

$$(12)\qquad \frac{r}{2} = -\frac{a^2}{\sqrt{2}\cdot V_\infty}\int_0^{\pi}\sqrt{\frac{1+\sin\sigma}{\cos\sigma}}\cdot(\cos\sigma - 1)\sin\sigma e^{-i\sigma}d\sigma =$$

$$= -\frac{a^2}{\sqrt{2}\cdot V_\infty}\int_0^{\frac{\pi}{2}} e^{+i\frac{\pi}{4}}\sqrt{\frac{1+\sin\sigma}{\cos\sigma}}\cdot(\cos\sigma - 1)\sin\sigma\, d\sigma -$$

$$-\frac{a^2}{\sqrt{2}\cdot V_\infty}\int_{\frac{\pi}{2}}^{\pi} e^{-i\frac{\pi}{4}}\sqrt{\frac{1+\sin\sigma}{\cos\sigma}}\cdot(\cos\sigma-1)\sin\sigma\, d\sigma$$

From relation (10) we will obtain for $\omega(\zeta)=\theta+i\tau$

(13) $$e^{i\omega(\zeta)}=e^{\frac{1}{2}\ln\frac{1-i\zeta}{1+i\zeta}}=\sqrt{\frac{1-i\zeta}{1+i\zeta}}=\sqrt{\frac{1-ie^{i\sigma}}{1+e^{i\sigma}}}=e^{-\tau}(\cos\theta+i\sin\theta)$$

and the constant $e^{-\tau}$ will be

(14) $$e^{-\tau}=e^{-i\theta}\cdot\sqrt{\frac{1-ie^{i\sigma}}{1+ie^{i\sigma}}}=e^{-i\theta}\sqrt{\frac{1+\sin\sigma}{\cos\sigma}}$$

4. THE CALCULATION OF THE TOTAL PRESSURE ON THE OUTLINE

It is known[2] that the resultant of the pressures on the outline is given by the formula [2]

(15) $$R_x+iR_y=-\frac{i\rho V_\infty}{2}\int_C e^{i\omega(\zeta)}\cdot\frac{df}{d\zeta}\,d\zeta$$

Using the fields for $e^{i\omega}$ from (13) and $\frac{df}{d\zeta}$ from (11) we have:

(16) $$\frac{df}{d\zeta}=-a^2\left[1-\frac{\zeta}{2}-\frac{1}{\zeta^2}+\frac{1}{2\zeta^3}\right]$$

$$\omega(\zeta)=\omega'(0)\zeta+\frac{1}{2}\omega''(0)\cdot\zeta^2+\ldots$$

$$e^{i\omega}=1+i\omega-\frac{\omega^2}{2!}-i\frac{\omega^3}{3!}+\ldots=1+i(-\zeta)+0=1-i\zeta$$

$$\frac{df}{d\zeta}e^{i\omega}=a^2(1-i\zeta)\left(-1+\frac{\zeta}{2}+\frac{1}{\zeta^2}-\frac{1}{2\zeta^3}\right)$$

$$(17) \qquad Rez\left\{\frac{df}{d\zeta}e^{i\omega}\right\} = -a^2 i$$

Replacing in (15) we obtain:

$$R_x = -\frac{\rho V_\infty}{2}a^2 \ ; \quad R_y = 0 \quad \text{and} \quad R = \frac{\rho V_\infty}{2}a^2$$

where a^2 - constant given by the relation (12)

The total pressure coefficient $C_x = \dfrac{R}{\frac{1}{2}\rho V_\infty^2 \cdot 2r} = \dfrac{a^2}{2r \cdot V_\infty}$

REMARKS

1. The considered model approximates also a plan configuration as in figure 9.

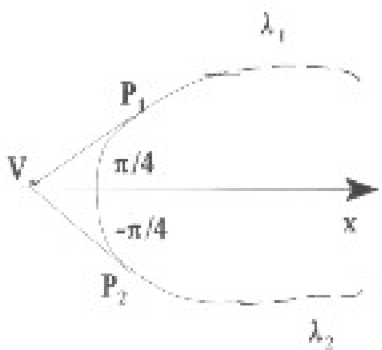

Figure 9

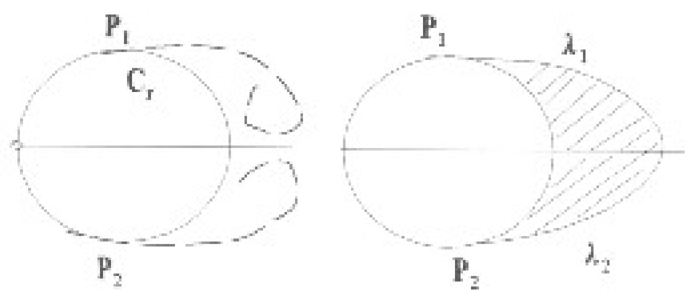

Figure 10

2. We wonder if lines λ_1 and λ_2 unite or give other sources of vortexes. Figure 10.

3. It is possible that the particle detachments take place after a temporal law of periodic type (pulsatile), so that in the area between the circles current lines of λ_1 and λ_2 form to be generated and also vortexes, the area being finite.

The conjugation of these facts in time may lead to the total of the pressure forces on the outline Cr with undesired results for a real case.

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