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Dedicated to Professor Ion PĂVĂLOIU on his 60^{th} anniversary

On the r-convergence orders of the inexact perturbed Newton methods

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Abstract

The inexact perturbed Newton methods recently introduced by us are variant of Newton method, which assume that at each step the linear systems are perturbed, and then they are only approximately solved.

The q-convergence orders of the iterates were characterized using the results of Dembo, Eisenstat and Steihaug on inexact Newton methods.

In this note we deduce, in the same manner, the characterization of the r-convergence orders of these iterates.

1 Introduction.

Given an open set $D \subseteq \mathbb{R}^n$ and $F : D \to \mathbb{R}^n$, the Newton method

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \quad k = 0, 1, \ldots, \quad x_0 \in D,$$

is a classical way which allows in certain circumstances the approximation of a solution x^* of the nonlinear system $F(x) = 0$.

Its local convergence is usually studied in the following hypotheses:

(C1) there exists $x^* \in \mathbb{R}^n$ such that $F(x^*) = 0$;

(C2) the mapping F is differentiable on a neighborhood of x^* and F' is continuous at x^*;

(C3) the Jacobian $F'(x^*)$ is nonsingular;

(C4) the derivative F' is Hölder continuous with exponent $p \in (0, 1]$ at x^*, i.e. for an arbitrary fixed norm $\|\cdot\|$ on $\mathbb{R}^n$ there exist $L, \varepsilon > 0$ such that

$$\|F'(x) - F'(x^*)\| \le \|x - x^*\|^p, \quad \text{when } \|x - x^*\| < \varepsilon.$$

Before enouncing the known convergence results concerning the Newton iterates, we briefly remind the definitions concerning the convergence orders of the sequences. An arbitrary sequence $(y_k)_{k\ge 0} \subset \mathbb{R}^n$ is said that converges q-superlinearly to its limit $\bar{y} \in \mathbb{R}^n$ if

$$\lim_{k\to\infty} \frac{\|y_{k+1} - \bar{y}\|}{\|y_k - \bar{y}\|} = 0, \quad \text{when } y_k \ne \bar{y},\ k \ge k_0,$$

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and with q-convergence order $\alpha > 1$ if

$$\limsup_{k\to\infty} \frac{\|y_{k+1} - \bar{y}\|}{\|y_k - \bar{y}\|^\alpha} < +\infty, \quad \text{when } y_k \neq \bar{y},\ k \geq k_0.$$

The sequence $(y_k)_{k\geq 0} \subset \mathbb{R}^n$ is said that converges r-superlinearly to its limit $\bar{y} \in \mathbb{R}^n$ if

$$\lim_{k\to\infty} \|y_k - \bar{y}\|^{1/k} = 0,$$

and with r-convergence order $\alpha > 1$ if

$$\limsup_{k\to\infty} \|y_k - \bar{y}\|^{1/\alpha^k} < 1, \quad \text{when } y_k \neq \bar{y},\ k \geq k_0.$$

These definitions are more rigorously treated in the classical book of Ortega and Rheinboldt [12, ch.9] (see also [15], [14]).

Conditions **(C1)–(C3)** assure an attraction theorem for the Newton iterates: there exists $\varepsilon > 0$ such that for any initial approximation $x_0 \in D$ with $\|x_0 - x^*\| < \varepsilon$, the Newton iterates $(x_k)_{k\geq 0}$ are well defined, remain in the ball of center x^* and radius ε, and converge q-superlinearly to x^* (see [12, Th. 10.2.2]). Under the additional condition **(C4)**, the convergence is with q-order $1 + p$.

The r-convergence orders are more general than the q-convergence orders, in the sense that a sequence converging q-superlinearly converges also r-superlinearly, and when it converges with q-order $\alpha > 1$, it also converges with r-order α (the converse being not true as a general affirmation). Therefore, in the conditions of the above attraction theorem, we automatically obtain r-superlinear convergence, resp. convergence with r-orders $1 + p$ of the Newton iterates.

2 The r-convergence orders of the inexact perturbed Newton methods.

When used in practice, the Newton method results in the following algorithm:

Choose an initial approximation $x_0 \in D$
For $k = 0, 1, \ldots$ until "convergence" do
  Solve $F'(x_k)\, s_k = -F(x_k)$
  Set $x_{k+1} = x_k + s_k$.

Each step requires the solving of a linear system, which is usually a difficult task. There exist two main approaches in order to overcome this difficulty. The first one considers some linear systems easier to solve, by perturbing the matrices; the resulting iterations are called quasi-Newton methods. The second approach considers that the linear systems are not solved exactly:

Choose an initial approximation $x_0 \in D$
For $k = 0, 1, \ldots$ until "convergence" do
  Find s_k such that $F'(x_k)\, s_k = -F(x_k) + r_k$
  Set $x_{k+1} = x_k + s_k$.

The error terms (the residuals) r_k represent the amounts by which the solutions s_k fail to satisfy the exact linear systems. Dembo, Eisenstat and Steihaug characterized the convergence orders of the inexact Newton (IN) method above. We remind the result dealing with the r-convergence orders.

Theorem 1 *[7] Assume that conditions (C1)–(C4) hold and the IN iterates $(x_k)_{k\geq 0}$ converge to x^*. Then the convergence is with r-order at least $1+p$ if and only if $r_k \to 0$ with r-convergence order $1+p$ as $k\to\infty$.*

We have recently proposed in [4] a new model for the Newton methods, which reflects the different situations that usually arise in practice:

$$\begin{aligned} &(F'(x_k)+\Delta_k)\, s_k = (-F(x_k)+\delta_k)+\tilde{r}_k \\ &x_{k+1} = x_k + s_k, \quad k=0,1,\ldots, \quad x_0 \in D. \end{aligned}$$

The matrices $(\Delta_k)_{k\geq 0} \subset \mathbb{R}^{n\times n}$ represent perturbations to the Jacobians, the vectors $(\delta_k)_{k\geq 0} \subset \mathbb{R}^n$ perturbations to the function evaluations for $-F(x_k)$, while $\tilde{r}_k$ are the residuals of the approximate solutions s_k of the perturbed linear systems $(F'(x_k)+\Delta_k)\, s = -F(x_k)+\delta_k$. These iterations were called inexact perturbed Newton (IPN) iterations, and in the mentioned paper we have characterized their q-convergence orders.

The r-convergence orders may also be characterized in the same manner:

Theorem 2 *Assume that conditions (C1)–(C4) hold, the matrices Δ_k are chosen such that the perturbed Jacobians $F'(x_k)+\Delta_k$ are invertible for $k=0,1,\ldots,$ and that the IPN iterates converge to x^*. Then the convergence is with r-order at least $1+p$ if and only if*

$$\left\| \Delta_k (F'(x_k)+\Delta_k)^{-1} F(x_k) + \left(I - \Delta_k (F'(x_k)+\Delta_k)^{-1}\right)(\delta_k + \tilde{r}_k) \right\| \to 0$$

with r-order $1+p$.

Proof. As for the q-convergence orders discussed in [4], the IPN can be viewed as an IN method:

$$\begin{aligned} s_k &= -(F'(x_k)+\Delta_k)^{-1} F(x_k) + (F'(x_k)+\Delta_k)^{-1}(\delta_k+\tilde{r}_k); \\ F'(x_k)\, s_k &= -\Delta_k s_k - F(x_k) + \delta_k + \tilde{r}_k \\ &= -F(x_k) + \Delta_k (F'(x_k)+\Delta_k)^{-1} F(x_k) - \\ &\quad \Delta_k (F'(x_k)+\Delta_k)^{-1}(\delta_k+\tilde{r}_k) + \delta_k + \tilde{r}_k \\ &= -F(x_k) + \Delta_k (F'(x_k)+\Delta_k)^{-1} F(x_k) + \\ &\quad \left(I - \Delta_k (F'(x_k)+\Delta_k)^{-1}\right)(\delta_k+\tilde{r}_k). \end{aligned}$$

Denoting

$$r_k = \Delta_k (F'(x_k)+\Delta_k)^{-1} F(x_k) + \left(I - \Delta_k (F'(x_k)+\Delta_k)^{-1}\right)(\delta_k+\tilde{r}_k),$$

the assertion follows from Theorem 1. □

It is interesting to note that there exist iterations with convergence orders to be analyzed and characterized, as we shall see in a forthcoming paper.

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