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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

A characterization of the efficient solutions of multiobjective optimization problems by means of a scalar problem

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Abstract

A useful way to characterize the efficient solutions of multiobjective optimization problems is to refer to the optimal solutions of a scalar optimization problem derived from the multiobjective problem. We use this approach to give some characterizations of the efficient solutions. We also present a sufficient condition for an optimal solution of the scalar problem to be an efficient solution of the multiobjective problem.

1 Introduction

A common technique in multiobjective optimization problems is to convert the problem into a scalar optimization problem. This can be done in a variety of ways, and the multiobjective programming literature contains a lot of examples in this sense (see [2], [3], [5], [7], [8]). Such an approach can be useful when we try to characterize the efficient solutions (or weakly efficient solutions) of multiobjective problems using the optimal solutions of the attached scalar problem. The weight method is a well known way of obtaining a scalar problem from a multiobjective optimization problem, and we shall use this approach to characterize the efficient solutions of a multiobjective maximization problem.

The purpose of the present paper is, on the one hand, to give some characterizations of the efficient solutions in terms of the optimal solutions of the

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scalar problem associated by the weight method and, on the other hand, to extend and supplement a known result in [4].

2 Definitions and properties

Consider the multiobjective optimization problem:

$$(MOP)\begin{cases} f(x) = (f_1(x), ..., f_n(x)) \to \max \\ x \in X \end{cases}$$

where $X \subseteq \mathbb{R}^p$ is nonempty and $f = (f_1, ..., f_n) : X \to \mathbb{R}^n$ is a vector-valued function.

Let $f = (f_1, ..., f_n)$. We use the following relations. For $a, b \in \mathbb{R}^n$

$a \geqq b$ if $a_k \geqq b_k$, for $k = 1, ..., n$;

$a \geq b$ if $a_k \geqq b_k$, for $k = 1, ..., n$ with at least one strict inequality;

$a > b$ if $a_k > b_k$, for $k = 1, ..., n$.

Definition 2.1 *A point $\bar{x} \in X$ is called*

a) an efficient solution *of (MOP) if there does not exist $y \in X$ such that $f(y) \geq f(x)$;*

b) a weakly efficient solution *of (MOP) if there does not exist $y \in X$ such that $f(y) > f(x)$;*

c) a local (resp. weakly) efficient solution *of (MOP) if there does not exist $y \in U \cap X$ such that $f(y) \geq f(x)$ (resp. $f(y) > f(x)$) for some neighborhood U of $\bar{x}$.*

We shall also need the definitions of some types of functions, similar to concave functions, but which have only some of the properties of those functions.

Definition 2.2 *Let $X \subseteq \mathbb{R}^p$ be a convex set and $g : X \to \mathbb{R}$ a function. We say that g is*

a) quasiconcave *on X if, for all $x, y \in X$,*

$$g(ty + (1-t)x) \geqq \min\{g(x), g(y)\}, \text{ for all } t \in]0, 1[;$$

b) strictly quasiconcave *on* X *if, for all* $x, y \in X$ *such that* $g(y) \neq g(x)$,

$$g(ty + (1-t)x) > \min\{g(x), g(y)\}, \text{ for all } t \in]0,1[;$$

c) explicitly quasiconcave *on* X *if* g *is strictly quasiconcave and quasiconcave;*

d) strongly quasiconcave *on* X *if, for all* $x, y \in X$ *such that* $x \neq y$,

$$g(ty + (1-t)x) > \min\{g(x), g(y)\}, \text{ for all } t \in]0,1[.$$

It is easy to see that the following implications hold: for a function f:

f concave $\Rightarrow$ f explicitly quasiconcave $\Rightarrow$ f quasiconcave and strictly quasiconcave.

However, there exist quasiconcave functions which are not strictly quasiconcave and reciprocally, as can be seen from the following examples.

Example 1. Consider $g : [-1,1] \to \mathbb{R}$,

$$g(x) = \begin{cases} -x, & x \in [-1,0] \\ 0, & x \in]0,1]. \end{cases}$$

It can be easily seen that g is a quasiconcave function but it is not strictly quasiconcave.

Example 2. Let $g : [-1,1] \to \mathbb{R}$,

$$g(x) = \begin{cases} -x, & x \in [-1,1[\\ 1, & x = 1. \end{cases}$$

In this case, g is a strictly quasiconcave function but it is not quasiconcave.

However, a strictly quasiconcave function which is upper-semicontinuous is always a quasiconcave function according to [1]. It is also worth noting that a strongly quasiconcave function is strictly quasiconcave; the converse is not true, since it suffices to consider $g : [-1,1] \to \mathbb{R}$ where

$$g(x) = \begin{cases} 0, & x \neq 0 \\ -1, & x = 0. \end{cases}$$

For other properties and relations between quasiconcave, strictly quasiconcave and strongly quasiconcave functions we refer to [9].

3 Necessary and sufficient conditions

For (MOP) we consider the scalar optimization problem:

$$(P(w)) \quad \begin{cases} \sum_{i=1}^{n} w_i f_i(x) \to \max \\ x \in X \end{cases}$$

where $w = (w_1, ..., w_n)$ is a weight vector from $\mathbb{R}^n$.

We mention in the beginning some known results concerning the characterization of efficient solutions of (MOP) in terms of the optimal solutions of the corresponding scalar problem $(P(w))$ ([4], [7]). The first two results provide necessary conditions for the efficient solutions, while the next two results are sufficient conditions for the efficient solutions of (MOP), which we shall complete with a new result in Theorem 3.2.

Proposition 3.1 *[7] If $X \subseteq \mathbb{R}^p$ is a convex set and $f : X \to \mathbb{R}^n$ is a concave function on X, then for each efficient solution $\bar{x}$ of (MOP), there is at least one vector $w \in W = \left\{ w = (w_1, ..., w_n) \mid w_i \geqq 0,\ i = \overline{1,n},\ \sum_{i=1}^{n} w_i = 1 \right\}$ such that $\bar{x}$ is an optimal solution of $(P(w))$.*

We shall consider the following assumption concerning the function f:
Assumption (A): For any $x \in X$ and $y \in X$, we have

$$\operatorname{Max} f(\lambda x + (1-\lambda) y) \geqq \{\lambda \operatorname{Max} f(x) + (1-\lambda) \operatorname{Max} f(y)\},$$

for all $\lambda \in]0,1[$, where $\operatorname{Max} f(x) = \max\{f_1(x), ..., f_n(x)\}$.

Proposition 3.2 *[4] Consider the problem (MOP), where $X \subseteq \mathbb{R}^p$ is a convex set and $f : X \to \mathbb{R}^n$ is a componentwise quasiconcave function. If $\bar{x} \in X$ is an efficient solution of (MOP) and if Assumption (A) holds, then there exists a nonzero $w \in \mathbb{R}^n_+ \setminus \{0\}$ such that*

$$w^T f(\bar{x}) \geqq w^T f(x), \quad \text{for all } x \in X,$$

which means that $\bar{x}$ is an optimal solution of $(P(w))$.

Proposition 3.3 *[7] The point $\bar{x} \in X$ is an efficient solution of the problem (MOP) if there exists a weight vector $w \in W$ such that $\bar{x}$ is an optimal solution of the optimization problem $(P(w))$ attached to (MOP) and if at least one of the following conditions hold:*
(i) $\bar{x}$ is a unique optimal solution of $(P(w))$;
(ii) $w_j > 0$ for each $j \in \{1, ..., n\}$.

Further, we recall some definitions concerning the vector-valued functions.

Let X be a convex set from $\mathbb{R}^p$. We say that $g = (g_1, ..., g_n) : X \to \mathbb{R}^n$ is (k)-*strongly quasiconcave* on X if its components g_i, $i \in \{1, ..., n\} \setminus \{k\}$, are quasiconcave on X and if g_k is a strongly quasiconcave function on X.

Luc and Schaible [6] have introduced for the vector-valued case the concept of explicit quasiconcavity, which is more general than componentwise explicit quasiconcavity. Namely considering $f = (f_1, ..., f_n) : X \to \mathbb{R}^n$, where $X \subseteq \mathbb{R}^p$ is a convex set, then f is called *explicitly quasiconcave* if f is componentwise quasiconcave and, for all $x, y \in X$ such that $f(y) \neq f(x)$, we have

$$f(ty + (1-t)x) \geq \mathrm{Min}(f(x), f(y)), \qquad \text{for all } t \in]0, 1[,$$

where $\mathrm{Min}\{f(x), f(y)\} = (\min\{f_1(x), f_1(y)\}, ..., \min\{f_n(x), f_n(y)\})^t \in \mathbb{R}^n$.

We give now a result connecting the multiobjective problem (MOP) and the scalar optimization problem $(P(w))$. We also include its proof since we refer to it later.

Theorem 3.1 *[4] Consider (MOP), where $X \subseteq \mathbb{R}^p$ is a convex set. For some $k \in \{1, ..., n\}$ fixed, let $f : X \to \mathbb{R}^n$ be (k)-strongly quasiconcave on X, $w \in \mathbb{R}^n_+$ with $w_k > 0$ and let $\bar{x} \in X$ be a* local *optimal solution of*

$$(P(w)) \quad \begin{cases} w^T f(x) \to \max \\ x \in X. \end{cases}$$

Then $\bar{x}$ is an efficient solution of (MOP).

Proof: Assume the contrary, i.e. $\bar{x}$ is an optimal solution of $(P(w))$ but is not an efficient solution of (MOP). Hence there exists some $x^0 \in X$ such that $f(x^0) \geq f(\bar{x})$. Then, $f_j(x^0) \geqq f_j(\bar{x})$, for all $j \in \{1, ..., n\}$ and $f(x^0) \neq f(\bar{x})$, which means $x^0 \neq \bar{x}$.

Then, for any $\lambda \in]0,1[$, we have

$$f_j\left(\lambda x^0 + (1-\lambda)\bar{x}\right) \geqq \min\left\{f_j\left(x^0\right), f_j\left(\bar{x}\right)\right\} = f_j\left(\bar{x}\right), \quad \text{for all } j \in \{1,...,n\}$$

and

$$f_k\left(\lambda x^0 + (1-\lambda)\bar{x}\right) > \min\left\{f_k\left(x^0\right), f_k\left(\bar{x}\right)\right\} = f_k\left(\bar{x}\right).$$

Hence,

$$\sum_{j=1}^{n} w_j f_j\left(\bar{x} + \lambda\left(x^0 - \bar{x}\right)\right) > \sum_{j=1}^{n} w_j f_j\left(\bar{x}\right), \quad \text{for all } \lambda \in]0,1[,$$

i.e.,

$$w^T f\left(\bar{x} + \lambda\left(x^0 - \bar{x}\right)\right) > w^T f\left(\bar{x}\right), \quad \text{for all } \lambda \in]0,1[.$$

For λ small enough, we have that in some neighborhood of $\bar{x}$, there exists a point $y \in X$ such that

$$w^T f\left(y\right) > w^T f\left(\bar{x}\right),$$

which contradicts the fact that $\bar{x}$ is a local solution of $(P(w))$. □

If the assumption of (k)-strongly quasiconcavity on f is replaced by the assumption of explicit quasiconcavity, we get a similar result.

Theorem 3.2 *Consider (MOP), where $X \subseteq \mathbb{R}^p$ is a convex set and $f : X \to \mathbb{R}^n$ is explicitly quasiconcave on X. Let $w \in \mathbb{R}^n$ with $w_i > 0$ for each $i \in \{1,...,n\}$. If $\bar{x} \in X$ is a* local *optimal solution of $(P(w))$, then $\bar{x}$ is an efficient solution of (MOP).*

Proof: Assume that $\bar{x} \in X$ is a local optimal solution of $(P(w))$ but it is not an efficient solution of (MOP), i.e. there exists some $x^0 \in X$ such that $f(x^0) \geq f(\bar{x})$. Then, $f(x^0) \neq f(\bar{x})$ and from the assumption of explicit quasiconcavity on f, we have that

$$f\left(\lambda x^0 + (1-\lambda)\bar{x}\right) \geq \text{Min}\left\{f\left(x^0\right), f\left(\bar{x}\right)\right\} = f\left(\bar{x}\right), \quad \text{for all } \lambda \in]0,1[.$$

Hence,

$$f_j\left(\lambda x^0 + (1-\lambda)\bar{x}\right) \geqq f_j\left(\bar{x}\right), \qquad \text{for all } j \in \{1,...,n\}$$

and there exists $k \in \{1, ..., n\}$ such that $f_k(\lambda x^0 + (1-\lambda)\bar{x}) > f_k(\bar{x})$. But $w > 0$, and then

$$\sum_{j=1}^{n} w_j f_j(\lambda x^0 + (1-\lambda)\bar{x}) > \sum_{j=1}^{n} w_j f_j(\bar{x}), \quad \text{for all } \lambda \in]0,1[.$$

Therefore for λ small enough, we have that $\bar{x}$ is not a local solution of $(P(w))$, which is a contradiction. □

We notice that Theorem 3.2 is more general than Theorem 3.1. Indeed, if $f : X \to \mathbb{R}^n$ is a (k)-strongly quasiconcave function then f is explicitly quasiconcave; the converse is not true, as we can see from the following example.

Example 3: Let $f = (f_1, f_2) : [-1,1] \to \mathbb{R}^2$ be a vector-valued function, where

$$f_1(x) = \begin{cases} -x, & x \in [-1,0] \\ 0, & x \in]0,1] \end{cases}$$

$$f_2(x) = \begin{cases} 0, & x \in [-1,0] \\ -x, & x \in]0,1]. \end{cases}$$

Now we give an example which shows that, although Theorem 3.1 cannot be applied, we can apply Theorem 3.2.

Example 4: Let $f : [-1,1] \to \mathbb{R}^2$, where

$$f_1(x) = \begin{cases} -x, & x \in [-1,0] \\ 0, & x \in]0,1] \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & x \in [-1,0[\\ \frac{1}{2}, & x = 0 \\ -x, & x \in]0,1]. \end{cases}$$

It is easy to check that f is explicitly quasiconcave but it is not (k)-strongly quasiconcave, for each $k = 1,2$, so that we are in the hypotheses of Theorem 3.2.

Define the multiobjective optimization problem

$$(MOP) \begin{cases} f(x) = (f_1(x), f_2(x)) \to \max \\ x \in [-1,1]. \end{cases}$$

Let $w = (1,1) \in \mathbb{R}^2_+$ be a weight vector. For the given problem (MOP), we attach the scalar optimization problem

$$(P(w)) \quad \begin{cases} g(x) = w_1 f_1(x) + w_2 f_2(x) \to \max \\ x \in [-1,1] \end{cases}$$

where obviously

$$g(x) = \begin{cases} -x, & x \neq 0 \\ \frac{1}{2}, & x = 0. \end{cases}$$

Then we see that $x_0 = -1$ is a maximum point of g on $[-1,1]$, and on the basis of Proposition 3.3 we get that x_0 is also an efficient solution of the problem (MOP), which can be easily checked. But $x_1 = 0$ is a local maximum point of $(P(w))$. We cannot use Theorem 3.1 in this case, but we can apply Theorem 3.2, which assures that x_1 is an efficient solution of (MOP).

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