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Dedicated to Professor Ion PĂVĂLOIU on his 60[th] anniversary

THE CIRCLE TOPOLOGY IN THE PLANE

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Abstract

We study the *circle topology* in the plane , which is somewhere between the *fine topology* from the potential theory (derived from the *logarithmic potential* in the plane) and the (ordinary) *density topology* in the plane. We show that this topology is not normal.

1. Introduction.

Studying the sense preserving mappings in the complex plane we introduced in [6] the sun, *circle* and *helm* topologies. The properties of the sun topology was studied in [7]. In the present paper we will prove some properties of the circle topology.

We recall the *circle topology* in the plane, which is somewhere between the *fine topology* from the potential theory (derived from the *logarithmic potential* in the plane) and the (ordinary) *density topology* in the plane (the density at any point x is measured using the Lebesgue measure λ and discs centered at x).

Definition 1.1. The *circle* topology in the complex plane $\mathbb{C}$ is defined by declaring a set A to be circle open if for each $x \in A$ there exists a set $D_x \subset \mathbb{R}_+$ such that $\{z \in \mathbb{C} : |x-z| \in D_x\} \subset A$ and the $0 \in \mathbb{R}$ is a point of (one-dimensional) right-sided density for the set D_x. In other words the set A contains with any point x circles (not discs) with radii $r \in D_x$ and the set $D_x \cup (-D_x)$ is density open at the origin.

We show that the circle topology in the complex plane is finer than the fine topology and coarser that the density topology. Moreover, we show that the circle topology is not normal.

For any topology (e.g. blue) we use the terms blue open, blue Borel set ... with respect to this topology.

2. Results.

The following proposition relates the circle topology to the density topology in the plane.

Proposition 2.1. *The density topology in the plane is strictly finer than the circle topology. Any circle open set is Lebesgue measurable.*

Proof. Let the set A be circle open at x. Then A is density open at x due to the set D_x of radii of circles in A ($D_x \cup (-D_x)$ is density open at the origin, when D_x reaches the right-sided linear density q at 0 then A reaches the two-dimensional density q^2 at x). Moreover, we see that A is measurable (the density open set).

We observe that the set $C \setminus R_+$ is density open at 0 and is not circle open at 0. The density topology is strictly finer than the circle topology.

Now we introduce the *fine topology* in the plane from the potential theory. If we denote for an arbitrary point $x \in \mathbb{C}$, a set $A \subset \mathbb{C}$ and $n \in \mathbb{N}$

$$A_n(x) = \{z \in C \setminus A, \frac{1}{2^n} \leq |z - x| < \frac{2}{2^n}\} \quad ,$$

we can characterize the points x at which the set A is finely open as those for which the series

$$\sum_{n=1}^{\infty} \frac{-n}{\log cap^* (A_n(x))}$$

converges (*Wiener's test*), where cap^* denotes the outer *logarithmic capacity* (see [8], Theorem 5.4.1).

We will recall the useful property of fine topology in the plane (see [2], Th.10.14).

Proposition 2.2. *Let $A \subset \mathbb{C}$ and let A be finely open at a point x. Then there exist arbitrarily small $r > 0$ such that*

$$\{z \in C, |x - z| = r\} \subset A \quad .$$

This statement holds in the plane. The fine topology does not have such a property in higher dimensions. We improve this statement in the next proposition relating the fine topology in the plane to the circle topology.

Proposition 2.3. *The circle topology is strictly finer than the fine topology in the plane.*

Proof. Let $0 \subset B$ and let B be finely open at 0. Let $A = \mathbb{C} \setminus P(C \setminus B)$, where P is the projection $z \mapsto |z|$ onto the positive real line. Then A is finely open at 0 due to Wiener's test and the fact that the outer logarithmic capacity is non-increasing under contractions. Now we employ Wiener's test for the set A, which reads

$$\sum_{n=1}^{\infty} \frac{-n}{log\, cap^* (A_n(x))} < \infty \quad .$$

We recall the inequality

$$\mathcal{H}_1(K) \leq 4(cap' K)$$

which holds for all bounded Borel subsets of the real line (see [8], Theorem 5.3.2, here $\mathcal{H}_1$ is the 1-dimensional Hausdorff measure). Now we use the regularity of the capacity and estimate the right-sided one-dimensional density of $R \setminus A$ at the origin.

We conclude that A is density open at 0. We know that B contains all circles with radii in $A \cap \mathbb{R}_+$. Hence A and B are circle open at the origin.

We observe that the set $C \setminus \{z \in C : 1/|z| \in N\}$ is obviously circle open at the origin and is not finely open at the origin due to Wiener's test (the excluded circles $\{z \in \mathbb{C} : 1/|z| = n\}$ have the logarithmic capacity $1/n$). The circle topology is strictly finer than the fine topology in the plane.

Let *blue* and *green* be two topologies on a space X. We say that the topology blue has the green G_δ - *insertion property* if for each blue open set $\mathcal{G}$ and each blue closed set $\mathcal{F}$ with $\mathcal{G} \subset \mathcal{F}$, there is a set G of type green G_δ such that $\mathcal{G} \subset G \subset \mathcal{F}$ ([3], pp. 39 - 40).

From [5], Theorem 2.2 we recall the following

Proposition 2.4. *Let the topology blue be finer than the topology green on X. Let the topology blue have the green G_δ - insertion property. Suppose $\mathcal{A}$ and $\mathcal{B}$ are disjoint blue closed, $\mathcal{U}$ and $\mathcal{V}$ are disjoint blue open, $\mathcal{A} \subset \mathcal{U}$ and $\mathcal{B} \subset \mathcal{V}$. Then there exist A and B of type green F_σ such that $\mathcal{A} \subset A$, $\mathcal{B} \subset B$, A is disjoint with $\mathcal{B}$ and B is disjoint with $\mathcal{A}$.*

Now we can prove

Proposition 2.5. *The circle topology is not normal.*

Proof. Let the circle topology be normal. Let C be the Cantor (middle thirds) set in $[0, 1]$. Then for each $a \subset C$, the sets a and $b = C \setminus a$ are (disjoint) circle closed sets (C is a Lebesgue null set). Normality implies that there are disjoint circle open sets A and B, $a \subset A$, $b \subset B$. The sets a and b are density closed and A and B are density open due to Proposition 2.1. The sets a and b are density separated. The density topology in the plane has the Euclidean G_δ - insertion property ([3], 6.A.10 for the linear case, the proof in the plane is the same). Hence Proposition 2.5 applies. There exists F of type Euclidean F_σ such that $a \subset F$, F is disjoint with b.

The mapping $a \mapsto F$ is an injective mapping (we see that $F \cap C = a$) from the potential set $\mathcal{P}(C)$ to the collection of all Euclidean Borel sets, hence the cardinality argument applies. The cardinality of the collection of all Euclidean Borel sets is c whereas the cardinality of $\mathcal{P}(C)$ equals 2^c - a contradiction. The circle topology is not normal.

The following problem for the circle topology remains open.

Question2.6*Is the circle topology (completely) regular?*

We prove two useful properties of the circle topology

Proposition 2.7. *The circle topology has the following properties:*

(i) the circle topology fulfills the essential radius condition, this means that for each $x \in \mathbb{C}$ and each circle neighborhood U of x there is an "essential radius" $r(x, U) > 0$ such that

$$|x - y| \leq min(r(x, U_x), r(y, U_y)) \Rightarrow U_x \cap U_y \neq \emptyset$$

for every circle neighborhoods U_x, U_y of x and y in $\mathbb{C}$,

(ii) the circle topology has the Euclidean G_δ - insertion property , this means that for each circle open set $\mathcal{G}$ and each circle closed set $\mathcal{F}$ with $\mathcal{G} \subset \mathcal{F}$, there is a set G of type Euclidean G_δ such that $\mathcal{G} \subset G \subset \mathcal{F}$,

Proof. Let U be a circle neighborhood of x. Due to the definition of the circle topology there exists $r(x, U) > 0$ such that for any $r \in (0, r(x, U))$ there exists r' such that $9r/10 < r' \leq r$ and $\{z \in \mathbb{C} : |x - z| = r'\} \subset$ U. Then $r(x, U)$ is the essential radius and the essential radius condition obviously holds. Statement (i) holds. Now (ii) holds due to (i) (see [3],2.D.16,p. 66).

Other material on this subject can be found in [3], [4], [5], [6], [7] and [9].

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