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Dedicated to Professor Ion PĂVĂLOIU on his 60^{th} anniversary

APPROXIMATION OF FUNCTIONS BY GENERALIZED RIESZ MEANS

Yuri L. NOSENKO

Zamansky-type results are obtained for generalized Riesz-means of Fourier series of integrable functions. More exactly, the deviation of such a function and its generalized Riesz means is represented with the help of improper integral of appropriate order difference for given function and the remainder. This remainder is estimated from above in terms of appropriate orders moduli of smoothness for given function.

1. INTRODUCTION

Let $f \in L_p$ $(1 \leq p \leq \infty)$ be a $2\pi-$ periodical function and let

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} A_k(x) \tag{1}$$

be the Fourier series for $f(x)$, let

$$\tilde{f}(x) \sim -i \sum_{k=-\infty}^{\infty} \text{sign } k A_k(x)$$

be the Fourier series for conjugate function $\tilde{f}$ for given function f.

Let

$$\Delta_\delta^s f(x) \quad \left(\Delta_\delta^s = \Delta^{\cdot}(\Delta_\delta^{s-1}), \Delta_\delta^1 f(x) = f\left(x + \frac{\delta}{2}\right) - f\left(x - \frac{\delta}{2}\right)\right)$$

be the symmetric difference of f of order $s \in N$ with the step δ at a point x and

$$\omega_s(f;h) = \Delta_{0<\delta\leq h} ||\Delta_\delta^s f(x)||$$

be the modulus of smoothness of $f(x)$ of order s and step h . Here and below all the norms are in the metric of the space L_p.

The generalized Riesz means for (1) are called the means

$$R_n^{s,\alpha}(f;x) = \sum_{|k|\le n} \left(1 - \left(\frac{|k|}{n+1}\right)^s\right)^\alpha A_k(x). \tag{2}$$

Both reductions of (2) $R_n^{s,1}$ and $R_n^{1,\alpha}$ are known as Riesz means while further reductions $R_n^{1,1} = \sigma_n$ are known as arithmetic means of (1).

The deviations $f(x) - R_n^{s,1}(f,x)$, $f(x) - R_n^{1,\alpha}(f;x)$, were investigated by some authors in different directions. One of these directions is so called Zamansky-type formula which represents these deviations as the improper integral of appropriate difference for $f(x)$ and the remainder with the estimation for this remainder in terms of moduli of smoothness for $f(x)$. Namely, classical result due to M. Zamansky for a subclass of the class of continuous functions is [1]

$$f(x) - \sigma_n(f;x) = -\frac{1}{2\pi}\int_1^\infty \Delta^{\cdot 2}_{\frac{t}{n+1}} f(x) t^{-2} dt + \tau_n(f;x),$$

where

$$||\tau_n(f;x)||_C \le B\omega_2\left(f;\frac{1}{n+1}\right)_C \tag{3}$$

for some $B > 0$ which is independent of f and n.

Zamansky-type results were obtained for different means with Riesz [2], Cesaro (C,α) [3], Borel [4], Euler [4,5] among them.

Another generalization of Zamansky-type results gives bilateral estimation for $\tau_n(f;x)$ in (3) from both above and below (exact order) either in terms of second order moduli of smoothness or in terms of higher order moduli (see [6] for the results and the references).

2. MAIN RESULTS

Now we will formulate the appropriate Zamansky-type results for generalized Riesz means (2). These results are given for odd and even indices s separately. Constants C depend on mentioned parameters or are absolute constants. They are different at different occurrences generally speaking.

THEOREM 1. For $f \in L_p$ $(1 \le p \le \infty)$, $\alpha > 1$ and odd $s > 1$ there exists positive constant C which depends at most on p, s, α , such that

$$\begin{aligned} f(x) - R_n^{s,\alpha}(f;x) &= -c^* \int_1^\infty \Delta^{\cdot s+1}_{\frac{t}{n+1}} f(x) t^{-s-1} dt + \\ \tau_n(f;x), ||\tau_n(f;x)|| &\le C\omega_{s+1}\left(f;\frac{1}{n+1}\right), \end{aligned} \tag{4}$$

where $c^* = (-1)^{\frac{s+3}{2}} \frac{\alpha}{2I_{s+1}}$, $\quad I_s = \int_0^\infty \sin^s v \frac{dv}{v^s}$.

THEOREM 2. *For $f \in L_p$ $(1 \le p \le \infty)$, $\alpha > 1$ and even s there exists a positive constant C which depends at most on p, s, α, such that*

$$\begin{aligned} f(x) - R_n^{s,\alpha}(f;x) &= -c^* \int_1^\infty \Delta^{s+1}_{\frac{t}{n+1}} \tilde{f}(x) t^{-s-1} dt + r(f;x), \|r_n(f,x)\| \le \\ &\le C\left(\omega_{s+2}\left(f; \frac{1}{n+1}\right) + n\omega_{s+2}\left(F^{-}; \frac{1}{n+1}\right)\right), \qquad (5) \end{aligned}$$

where F^{-} is the conjugate function for the primitive F of f and $c^* = (-1)^{\frac{s}{2}+1} \frac{\alpha}{2I_{s+1}}$.

3. COMMENTARIES AND REMARKS

1. For Riesz means $R_n^{1,\alpha}(f;x)$ Zamansky-type result was obtained in the form (3) with the remainder estimated from both above and below. More exactly, there are positive constants C_1, C_2 being independent of f and n such that

$$C_1\omega_2\left(f; \frac{1}{n+1}\right) \le \|r_n(f;x)\| \le C_2\omega_2\left(f; \frac{1}{n+1}\right).$$

Theorems 1 and 2 give the estimate for the norm of remainder $\|r_n(f;x)\|$ in case of generalized Riesz means only from above. It is possible to prove that the estimate from below in this case is impossible in terms of the moduli of smoothness for $f(x)$ of the same orders as above.

2. Zamansky-type results are different for generalized Riesz means in cases of odd and even values of indices s.

3. Every improper integral over $[1, \infty)$ can be rewritten over the interwal $[\lambda, \infty)$. The results remain the same and constants C will be dependent on λ among other parameters.

4. PROOFS

First we will formulate some auxiliary results and will introduce some auxiliary notations which will be needed for the proofs of theorems.

The proofs of theorems 1 and 2 are based on the comparison principle for linear means of Fourier series proposed by R.M. Trigub (see [8]), classical results on the approximation properties of Riesz means, and the appropriate theorems on multiplicators.

Let us start with the comparison principle for linear means of (1). Given a matrix $\Lambda = \|\lambda_k^{(n)}\|$, whose elements depend on n $(k \in Z, n \in N)$ we will form the linear means (operators) for (1) with the help of this matrix

$$\tau_n(f;\Lambda) = \tau_n(f;\Lambda,x) \sim \sum_{k=-\infty}^{\infty} \lambda_k^{(n)} A_k(x)$$

and we suppose that the function $\tau_n(f;\Lambda,x)$ with given Fourier series belongs to L_p.

The norm $\tau(\Lambda)$ of this operator is known as Lebesgue constant for the corresponding matrix.

With the help of another matrix $\tilde{\Lambda} = \|\tilde{\lambda}_k^{(n)}\|$ we will form another means of (1)

$$\tau_n(f;\tilde{\Lambda}) = \tau_n(f;\tilde{\Lambda},x) \sim \sum_{k=-\infty}^{\infty} \tilde{\lambda}_k^{(n)} A_k(x)$$

and let the function $\tau_n(f;\tilde{\Lambda},x)$ belongs to L_p too.

THEOREM (Comparison principle by R.M. Trigub) *In the notations above (for f,Λ,τ) the following inequality holds*

$$\|\tau_n(f;\Lambda,x)\| \leq inf\ \tau(\Lambda^*)\|\tau_n(f;\tilde{\Lambda},x)\| \tag{6}$$

where $\tau(\Lambda^*)$ is the norm of the operator defined by the martix $\Lambda^* = \|\lambda_k^{*(n)}\|$, $\lambda_k^{*(n)} = \frac{\lambda_k^{(n)}}{\tilde{\lambda}_k^{(n)}}$, inf (infimum) is taken for the values of fractions $\frac{0}{0}$ if there are such fractions.

The matrix $\Lambda^*$is known as the transitional matrix for the above inequality.

Starting the proofs of theorems 1 and 2 we will introduce the notations

$$I_s = \int_0^\infty sin^s v \frac{dv}{v^s},\ i_s(u) = \int_0^1 sin^s \frac{ut}{t^s} dt.$$

The representation

$$\Delta_h^m f(x) \sim (2i)^m \sum_k sin^m \frac{kh}{2} A_k$$

is well known.

Using this representation and changing the variables in the improper integral we have

$$\int_1^\infty \Delta_{t/(n+1)}^{s+1} f(x) t^{-s-1} dt =$$
$$(-1)^{(s+1)/2} 2^{s+1} \sum_{k=-\infty}^{\infty} \left(\frac{k^s}{(n+1)^s 2^s} I_{s+1} - i_{s+1} \frac{k}{n+1}) \right) A_k(x).$$

To obtain (4) (theorem 1, sis odd and sis greater then 1) we will prove that

$$\|f(x) - R_n^{s,\alpha}(f;x) + c^* \int_1^\infty \Delta_{\frac{t}{n+1}}^{s+1} f(x) t^{-s-1} dt\| \leq C\|f(x) - R_n^{s,1}(f;x)\| \tag{7}$$

with the constant Cbeing independent of $f(x)$and n. Then it remains only to use well-known classical estimate (from above) for the right member of (7) in terms of moduli of smoothness for $f(x)$.

But the right member of (7) can be represented (without the norm-symbol and with c^*as in the statement of the theorem)

$$\sum_{|k|\leq n}\left(1-\left(1-\left(\frac{k}{n-1}\right)^s\right)^\alpha + c^{'}(-1)^{\frac{s+1}{2}}2^{s+1}\left(\frac{k^s}{(n+1)^s2^s}I_{s+1}-i_{s+1}(\frac{k}{n+1})\right)\right)\cdot$$

$$\cdot A_k(x)+\sum_{|k|\geq n}\left(1+c^*(-1)^{\frac{s+1}{2}}2^{s+1}\left((\frac{k^s}{(n+1)^s2^s}I_{s+1}-i_{s+1}(\frac{k}{n+1})\right)\right)A_k(x).$$

The expression $k/(n+1)$ here is the same as $|k|/(n+1)$. The matrix defining these means is the value of the function $\Lambda(u)$ with $u=|k|/(n+1)$ where

$$\Lambda(u)=1-(1-u^s)\alpha+c^*(-1)^{\frac{s+1}{2}}2u^sI_{s+1}-c^*(-1)^{\frac{s+1}{2}}2^{s+1}i_{s+1}(u), u\in[0,1],$$

$$\Lambda(u)=1-\frac{\alpha u^s}{I_{s+1}}\int_{u/2}^{\infty}sin^{s+1}v\frac{dv}{v^s+1}, u\in(1,\infty).$$

The correspondent function defining appropriate means in the right member of (7) is $\tilde{\Lambda}(u)=u^{s+1}$ and $\tilde{\Lambda}(u)=1$ for $u\in[0,1]$ and $u\in(1,\infty)$ respectively.

In accordance with (6) the transitional function for (7) is (transitional matrix can be obtained from $\Lambda(u)$ putting $u=\frac{|k|}{n+1}$)

$$\Lambda^*(u)=u^{-s-1}\left(1-(1-u^s)^\alpha+c^*(-1)^{\frac{s+1}{2}}u^sI_{s+1}-c^*(-1)^{\frac{s+1}{2}}2^{s+1}i_{s+1}(u)\right),$$

$$\Lambda^*(u)=1-\frac{\alpha u^s}{I_{s+1}}\int_{u/2}^{\infty}sin^{s+1}v\frac{dv}{v^{s+1}}$$

for $u\in[0,1]$ and $u\in(1,\infty)$ respectively and $\Lambda^*(0)=-\frac{\alpha}{2I_s+1}$ (we will complete the definition of $\Lambda(u)$ puting $\Lambda^*(0)=\lim\Lambda(u)$ provided u ends to 0).

It is easy to check that $\Lambda^*(u)$ is continuous and bounded over $[0,\infty)$. Some additional elements of behaviour of $\Lambda^*(u)$ are needed to be used to check that Λ^*belongs to $B(R)$(class of Fourier transforms of measures bounded over R) (see [8, 9]). This authomatically means (taking in account (6)) that (4) is valid.

Then it remains to use an estimate from above for $\|f(x)-R_n^{s+1,1}(f;x)\|$for even indices (here $s+1$is even).

Theorem 1 is proved.

Starting the proof of theorem 2 we will use the same notations as above with c^*this time as in theorem 2.

To prove (5) the inequality is needed to be proved ($s$is even)

$$\|f(x) - R_n^{s,\alpha}(f;x) + c^* \int_1^\infty \Delta_{t/(n+1)}^{s+1} \tilde{f}(x) t^{-s-1} dt\| \leq \|f(x) - R_n^{s,1}(f;x)\|$$

Transitional function for needed inequality in accordance with (6) is the function

$$\Lambda^*(u) = u^{-s-1}\left(1 - (1-u^s)^\alpha + 2c^*(-1)^{s/2} u^s I_{s+1} - c^* 2^{s+1} (-1)^{s/2} i_{s+1}(u)\right),$$

$$\Lambda^*(u) = 1 - \frac{\alpha u^s}{I_{s+1}} \int_{u/2}^\infty sin^{s+1} v \frac{dv}{v^{s+1}}$$

for $u \in [0,1]$and $u \in (1,\infty)$and $\Lambda^*(0)$as in theorem 1.

Arguing in the same manner as in the case of theorem 1 we will conclude the proof of theorem 2.

For the inverse inequalities for the remainders in (4), (5) transitional matrix with the elements $1/\lambda_k^{(n)}$are needed where $\lambda_k^{(n)}$are the elements of transitional matrix for proved inequalities. But the function $\Lambda(u)$in both cases (theorems 1 and 2) takes on zero values inside $[0,\infty)$. Lebesgue constants for such the matrices are unbounded. Inverse inequalities in (4), (5) for the remainders (from below) are impossible.

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Donetsk State Technical University, Department of Mathematics, Artema 58, Donetsk 340000, Ukraine
Yuri@nosenko.dgtu.donetsk.ua

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