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Dedicated to Professor Ion PĂVĂLOIU on his 60th anniversary

A TOPOLOGICAL DEGREE FOR A*-PROPER MAPPINGS ACTING FROM A BANACH SPACE INTO ITS DUAL

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Abstract. In this paper we define a topological degree for a class of operators $T: X \to X^*$, where X is a separable Banach space. This class generalizes the class of mappings of type (S_+). We define a projectionally complete scheme for (X, X*). We use the Browder-Petryshyn method and Galerkin approximations.

1. INTRODUCTION

Definition 1. Let X and Y be two separable Banach spaces, $(X_n)_{n\in N}$, $(Y_n)_{n\in N}$ be two increasing sequences of finite dimensional subspaces of X, respectively Y and $P_n: X \to X_n$, $Q_n: Y \to Y_n$, $n \in \mathbf{N}$ linear and continuous projections. Then $\Gamma := \{ X_n, P_n, Y_n, Q_n \}$ is a projectionally complete scheme for (X, Y) if $\dim X_n = \dim Y_n$, $P_n x \to x$ in X and $Q_n y \to y$ in Y, for each $x \in X$, $y \in Y$.

Let us consider a Banach space X such that X and X* are separable. We suppose that there exists an increasing sequence $(X_n)_{n\in N}$ of finite dimensional subspaces of X and let $P_n: X \to X_n$ be the corresponding projections.
We define $P_n^*: X^* \to X_n^*$ by

$$(1) \qquad \langle P_n^* x^*, x \rangle = \langle x^*, x \rangle,\ x^* \in X^*,\ x \in X_n.$$

We give the following:

Lemma 1. $\Gamma = \{ X_n, P_n, X_n^*, P_n^* \}$ is a projectionally complete scheme for (X,X*).

Indeed, it results from (1) that P_n^* are linear, continuous and idempotent.

Example. If X has complete system of linear independent elements (Schauder basis) denoted by $\{ e_1, e_2, \ldots, e_n, \ldots \}$, then it can be chosen $X_n = sp\{ e_1, e_2, \ldots, e_n \}$ and for $x = \sum_{j=1}^{\infty} x_j e_j$, define $P_n x = \sum_{j=1}^{n} x_j e_j$.

Definition 2. An application $T: \bar{D} \subset X \to X^*$ is (weak) A*-proper with respect the scheme $\Gamma = \{ X_n, P_n, X_n^*, P_n^* \}$ if the restrictions $T_n := P_n^*$

$T: X \to X_n^*$ are continuous for each $n \in \mathbf{N}$ and if Γ_m is a subscheme of Γ and $\{ x_m / x_m \in \overline{D} \cap X_m \}$ is a bounded sequence with $T_m(X_m) \to g$ in X^*, then there exist a subsequence $\{ x_i \} \subset \{ x_m \}$ and $x \in \overline{D}$ such that $x_i \to x$ (respectively $x_i \to x$) in X and $T(x)=g$.

Lemma 2. If $D \subseteq X$ is a bounded, open and $T: \overline{D} \to X^*$ is continuous and A*-proper, then T is proper, in particular T is a closed operator.

Proof. Let $K \subset X^*$ be compact, $M \subset D$ be closed and $\{x_k\} \subset M \cap T^{-1}(K)$. T is a continuous and Γ is a projectionally complete scheme, so far each $k \geq 1$ and $\delta_k := \frac{1}{k}$, there exist $n(k) \in \mathbf{N}$, $n(k) > k$ such that

$$\| x_k - z_{n(k)} \| < \delta_k, \quad z_{n(k)} := P_{n(k)} x_k \in X_{n(k)} \cap \overline{D}$$
$$\text{and } \| T(x_k) - T(z_{n(k)}) \| \leq \varepsilon_k \to 0, \text{ for } k \to \infty.$$

Then $P_{n(k)}^*(T(z_{n(k)})) \to g$ and because T is A-proper, there exist a subsequence $\{ z_{n(i)} \} \leq \{ z_{n(k)} \}$ such that $z_{n(i)} \to x$ in X and $T(x)=g$. Finally $x_i \to x$ in X, that is $M \cap T^{-1}(K)$ is compact.

Lemma 3. If $T: \overline{D} \subset X \to X^*$ is A*-proper and $p \in X^* - T(\partial D)$, then there exist $n_0 \geq 1$ and $b>0$ such that $\| T_n(x_n) - P_n^* p \| \geq b$, $\forall x_n \in \partial D_n$, $n \geq n_0$, where $\overline{D}_n = P_n(\overline{D})$.

Proof. $D_n \cap \partial D_n = \emptyset$ and $P_n(\overline{D})$ is closed. Therefore $\partial D_n \subseteq P_n(\partial D)$. Let us suppose that there exist $\varepsilon_j \to 0$, $\varepsilon_j > 0$ and a sequence $\{ x_j / _j \in \partial D_j \}$ such that $\| T_j(x_j) - P_j^* p \| \leq \varepsilon_j$. Now we can choose a subsequence $\{ x_i \} \subseteq \{ x_j \}$ and $x \in D$ with $P_i x_i \to x$ in X and $T(x)=p$, because T is A-proper. But $P_i x_i \in \partial D_i \subset \partial D \Rightarrow x \in \partial D$, contradiction.

The class of A*-proper operators is larger then some classes of mappings of monotone type. We say that $T:X \to X^*$ is of class (S_+) if for each sequence $\{ x_n \} \subset X$ with $x_n \to x$ and $\limsup \langle T(x_n) - T(x), x_n - x \rangle \leq 0$ it results $x_n \to x$.

Theorem 1. *Assume that X is a reflexive Banach space. Then each operator $T: \overline{D} \subset X \to X^*$ of type (S_+) is A*-proper.*

Proof. Let $\{x_n / x_n \in X_n\}$ be a bounded sequence such that $T_n(x_n) \to g$. We can suppose that $x_n \rightharpoonup x$ because X is reflexive. Then $\langle T(x), x_n - x \rangle \to 0$ and from $T(x_n) \to g$ and $x_n \rightharpoonup x$ it results $\langle T(x_n), x_n - x \rangle \to 0$.

Hence $\limsup \langle T(x_n) - T(x), x_n - x \rangle = 0$ and $x_n \to x$, because T is of type (S_+). From the continuity of T it results $T(x)=g$, which means that T is A*-proper map.

Let us consider a Banach space X such that X and X* are separable. Let $\Gamma = \{X_n, P_n, X_n^*, P_n^*\}$ be a projectionally complete scheme and $T: \overline{D} \subset X \to X^*$ be an A-proper map. We denote $\overline{D}_n = P_n(\overline{D})$ and suppose that D_n is open and bounded, for every $n \in \mathbf{N}$. If $p \notin T(\partial D)$ there exist n_0 such that $P_n^* p \notin T_n(\partial D)$, $\forall n \geq n_0$, where $T_n = P_n^* T : X_n \to X_n^*$. T_n are continuous, so we can consider the sequence $\{d_n\} \subset \mathbf{Z}$, where $d_n := d(T_n, D_n, P_n^* p)$ is the Browner topological degree, which is well defined for $n \geq n_0$ because $P_n^* p \notin T_n(\partial D_n)$ as we can see from lemma 3.

Definition 3. We define the generalized topological degree of the A*-proper map T in p relative to D, with respect to scheme Γ, be the next subset of

$\overline{Z} = Z \cup \{\pm\infty\}$.

$D^*(T,D,p) := \{ d \in \overline{Z} \ / \ \exists \{d_i\} \subset \{d_n\},\ d_i \to d \}$

2. THE PROPERTIES OF DEGREE FOR A*-PROPER MAPS

(I) (The solution propriety). If $D^*(T, D, p) \neq \{0\}$ then the eqoation $T(x)=p$ has solutions in D.

Proof. There exists a sequence $\{d_i\}$ with $d_i = d(T_i, D_i, P_i^* p) \neq 0$. From the propertis of Brouwer degree there exists $x_i \in D_i$ such that $T_i(x_i) = P_i^*$
But $P_i^* p \to p \Rightarrow T_i(x_i) \to p$ and we can choose a subsequence $\{x_j\} \subset \{x_i\}$ with $x_j \to x \in \overline{D}$ and $T(x)=p$, because T is A-proper. Moreover, $x \in D$ because $p \notin T(\partial D)$.

(II) (The invariance to homotopy). Let $H: \overline{D} \times [0,1] \to X^*$ be continuous in t, uniformly in $x \in \overline{D}$ such that $H(\cdot,t)$ is A*-proper, $\forall t \in [0,1]$.

If $p \in X^* - H(\partial D, [0,1])$ then $D^*(H(\cdot,t), D, p)$ is independent in $t \in [0,1]$ and consequence.

Proof. $D^*(H(\cdot,t), D, p)$ is well defined for each $t \in [0,1]$ because $p \notin H(\cdot,t)(\partial D)$. Let us suppose that for $n \geq n_0$ we have $P_n^* p \notin H(\cdot,t)(\partial D_n)$ so $d(P_n^* H(\cdot,t), D_n, P_n^* p)$ is well defined for $n \geq n_0$ and independent in $t \in [0,1]$ and consequently $D^*(H(\cdot,t), D, p)$ is independent in $t \in [0,1]$.

In particular the degree for A*-proper maps is independent under translations.

(III) (Continuity with respect the function). There exist $r = r(T, p) > 0$ such that $D^*(T, D, p) = D^*(S, D, p)$ for every A*-proper map S with $\sup\{\|T(x) - S(x)\| / x \in \overline{D}\} < r$.

Proof. For some n_0 we have $\| T_n(x) - P_n^*(x)p \| \geq b$, $\forall x \in \partial D_n$, $n \geq n_0$.

If $r < \frac{b}{a}$, where $a = \sup\{\| P_n^* x \|, n \in \mathbf{N}\}$, we define

$H_n(t,x) = T_n(x) + t(T_n(x) - S_n(x))$ and $\| H_n(t,x) \| \geq b - ar > 0$, so $d(T_n, D_n, P_n{}^*p) = d(S_n, D_n, P_n{}^*p)$ $\forall n \geq n_0$ which is $D^*(T, D, p) = D^*(S, D, p)$.

(IV) (Continuity with respect the point). For p, q in the same conex connected set of $X^* - T(\partial D)$ we have $D^*(T, D, p) = D^*(T, D, q)$.

Proof. There exists $c:[0,1] \to X^* - T(\partial D)$ continuous, $c(0) = p$, $c(1) = q$. The conclusion follows from the invariance of the degree to the homotopy $H(\cdot,t) = T - c(t)$.

(V). (The additivity with respect the domain) Let D^1, D^2 be open, bounded, $D \supset D^1 \cup D^2$, $\overline{D} = \overline{D^1} \cup \overline{D^2}, D^1 \cap D^2 = \Phi$ and $p \notin T(\partial D^1 \cup \partial D^2)$.

Then $D^*(T,D,p) \subseteq D^*(T,D^1,p) + {}^*(T,D^2,p)$.

Proof. $P_n^* p \notin T(\partial D_n^1 \cup \partial D_n^2), D_n^1 \cap D_n^2 = \Phi, \forall n > n_0$.

Then from the additivity of the Brouwer degree with respect the domain, we have $d_n := d(T_n, D_n, P_n^* p) = d_n^1 + d_n^2$, where $d_n^i = d(T_n, D_n^i, P_n^i p)$, $i = 1, 2$.

If $d \in D^*(T,D,p)$ then let $d_j \to d$.

Case 1. If $d_j^1 = k \in \mathbf{Z}$ then $d_j^2 \to d - k =: m \in D^*(T,D^2,p)$ and so $d = k + m$, with $k \in D^*(T,D^1,p)$, $m \in D^*(T,D^1,p)$.

Case 2. If $d_j^1 \to \pm\infty$ and $|d| < \infty$, then $d_j^2 \to \mp\infty$ and $d = \pm\infty(\mp\infty) \in D^*(T,D^1,p) + D^*(T,D^2,p)$, by convention.

Case 3. If $d_j^1 \to \pm\infty$ and $d = \pm\infty$, then $d_j^2 \to m$ with $|m| < \infty$ or $m = \pm\infty$ and $d \in D^*(T,D^1,p) + D^*(T,D^2,p)$.

(VI) (Dependence from the values on the border). If $T,S: D \to X^*$ are A^*-proper maps with T = S on ∂D and $p \notin T(\partial D)$, then $D^*(T, D, p) = D^*(S, D, p)$.

Indeed, it results from the invariance to linear homotopy $H(\cdot,t) = (1-t)T + tS$.

Finally we give an result analogue to Borsuk theorem:

Theorem 2. *Assume that $D \subset X$ is a symmetric domain, $0 \in D$ and $T: \overline{D} \to X^*$ is A^*-proper, odd and $0 \notin T(\partial D)$. Then $D^*(T, D, 0)$ contains only odd integers, eventually $\pm\infty$.*

Indeed, if $d \in D^*(T,D,0), d \in \mathbf{Z}$, then let $d_i \to d, d_i = d(T_i, D_i, 0)$ and the conclusion follows from the fact that d_i are odd, according to Borsuk theorem for Brouwer degree.

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