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B-CONTINUOUS FUNCTIONS OF n-VARIABLES

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Abstract. The notion of B-continuous function was introduced by K. Bögel [3],[4] and studied by M. Nicolescu [7], E. Dobrescu [6], I. Badea [1].

In the present paper one extend the notions of B-continuity and uniform B-continuity to the case of n-variate functions. Then, one establishes an important property of the n-variate difference operator.

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1. Preliminaries

Let $I \subseteq R$ be an interval and let R^{I^2} be the space of all bivariate real-valued functions defined on the square I^2.

Definition 1.1.[3] The operator $\Delta_2 : R^{I^2} \to R^{I^2}$

$$\Delta_2[f;x,y;s,t] = f(s,t) - f(s,y) - f(x,t) + f(x,y)$$

defined for any function $f \in R^{I^2}$ and any points $M(x,y), M(s,t) \in I^2$ is called operator of "bivariate difference" or, simple, " bivariate difference". It is well known that Δ_2 is a linear operator.

Definition 1.2. [3] The function $f \in R^{I^2}$ is B-continuous in the point $M(x,y) \in I^2$ if and only if $\lim\limits_{(s,t)\to(x,y)} \Delta_2[f;x,y;s,t] = 0$. If *f* is B-continuous in any point $M(x,y) \in I^2$, then f is B-continuous on I^2. The space of all B-continuous functions on I^2 is denoted by $C_b(I^2)$.

Definition 1.3 [7] The function $f \in R^{I^2}$ is B-bounded on I^2 if there exists a positive constant K so that $|\Delta_2[f;x,y;s,t]| \le K$, for any points $M(x,y), M'(s,t) \in I^2$.

Theorem 1.1 [7] *If $I^2 \subseteq R^2$ is a compact set and $f \in C_b(I^2)$, then f is B-bounded on I^2.*

Definition 1.4 [3] The function $f \in R^{I^2}$ is uniform B-continuous on I^2 if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ so that for any $(x,y),(s,t) \in I^2$ satisfying the conditions $|x-s| < \delta, |y-t| < \delta$, the inequality

$$|\Delta_2[f;x,y;s,t]| < \varepsilon \text{ holds.}$$

2. Main results

If $I \subseteq R$ is a bounded interval $f \in R^I$ let us denote

(2.1.) $$\Delta[f;M,M'] = \Delta_s[f;x_1] = f(s_1) - f(x_1)$$

for any points $M(x_1), M'(s_1) \in I$. Let now $f \in R^{I^2}$ be a bivariate function and let ${}_{s_1}\Delta$, ${}_{s_2}\Delta$ be the parametric extensions of the operator (2.1), i.e.:

(2.2) $${}_{s_1}\Delta[f;M,M'] = f(s_1,x_2) - f(x_1,x_2)$$

(2.3) $${}_{s_2}\Delta[f;M,M'] = f(x_1,s_2) - f(x_1 x_2)$$

for any points $M(x_1,x_2), M'(s_1,s_2)$.

It is immediately that the bivariate difference of $f \in R^{I^2}$ is the product ("tensorial product") of the parametric extension (2.2) and (2.3), i.e.

$\Delta_2[f;M,M'] = ({}_{s_1}\Delta \circ {}_{s_2}\Delta)[f;x_1,x_2]$, for any $M(x_1,x_2), M'(s_1,s_2) \in I^2$.

Definition 2.1. Let $I \subseteq R$ a bounded interval,

$I^n = I \times I \times \cdots \times I$, $f \in R^{I^n}$ and let $s_1\Delta, \cdots s_n\Delta$ be the parametric extensions of the operator (2.1). The operator $\Delta_n : R^{I^n} \to R^{I^n}$,

(2.4) $$\Delta_n[f;MM'] = \Delta_{s_1,\ldots,s_n}[f;x_1,\ldots x_n] = \left({}_{s_1}\Delta \circ \cdots \circ {}_{s_n}\Delta\right)(f;x_1 \cdots x_1)$$

is called "n-variate difference". In (2.4), $M(x_1,\ldots,x_n), M'(s_1,\ldots,s_n) \in I^n$.

Remark 2.1. It is immediately the following representation

(2.5) $$\Delta s_1,\ldots,s_n[f;x_1,\ldots,x_n] =$$

$$= f(s_1,\ldots,s_n) - \sum_{i=1}^{n} f(x_1,\ldots x_{i-1}, s_i,\ldots,x_n) +$$

$$+\sum_{\substack{i,j=1\\ i\neq j}} f\left(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots,x_{j-1},s_j,x_{j+1},\ldots,x_n\right)+\cdots+(-1)^n f(x_1,\ldots,x_n)$$

for any $f\in R^{I^n}$ and any points $M(x_1,\ldots,x_n),M'(s_1,\ldots,s_n)\in I^n$.

For $n=2$, (2.5) reduces to the classical bidimensional difference operator.

Definition 2.2. The function $f\in R^{I^n}$ is B-continuous in $M(x_1,\ldots,x_n)\in I^n$ if the equality:

(2.5) $\lim_{(s_1,\ldots,s_n)\to(x_1,\ldots,x_n)} \Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_n\right]=0$ holds.

If the function f is B-continuous in any $M(x_1,\ldots,x_n)\in I^n$, then f is B-continuous on I^n. The set of all B-continuous functions on I^n is denoted by $C_b(I^n)$.

Lemma 2.1. If $f\in C_b(I^n)$, then $g\in R^{I^n}$ defined by

(2.7) $$g(s_1,\ldots,s_n)=\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots x_n\right]+f(x_1\ldots x_n)$$

is continuous on I^n.

Proof. Clearly, $g(x_1,\ldots,x_n)=f(x_1,\ldots x_n)$, for any $(x_1,\ldots,x_n)\in I^n$.

Next

$$\lim_{(s_1,\ldots,s_n)\to(x_1,\ldots,x_n)} gs_1,\ldots,s_n=$$

$$=f(x_1,\ldots,x_n)+\lim_{(s_1,\ldots,s_n)\to(x_1,\ldots,x_n)}\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_n\right]$$

$=f(x_1,\ldots,x_n)$, because $\lim_{(s_1,\ldots,s_n)\to(x_1,\ldots,x_n)}\Delta_{s_1\ldots s_n}\left[f;x_1,\ldots,x_n\right]=0$,

f being B-continuous on I^n. It follows that g is continuous in every point $(x_1,\ldots x_n)\in I^n$ and the proof ends.

Definition 2.3. The function $f\in R^{I^n}$ is B-bounded on I^n if there exists a positive number K>0 so that

(2.8) $$\left|\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_n\right]\right|\le K$$

for any $M(x_1,\ldots,x_n),M'(s_1,\ldots,s_n)\in I^n$.

Lemma 2.2. If $I\subset R$ is bounded and $f\in C_b(I^n)$, then f is B-bounded I^n.

Proof. Immediately, by using the lemma 2.2.

Definition 2.4. The function $f \in R^{I^n}$ is uniform B-continuous on I^n if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ so that for any $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ satisfying the conditions $|s_i - x_i| < \delta$ $\left(i = \overline{1,n}\right)$ the inequality

(2.9) $$\left|\Delta s_1,\ldots,s_n\left[f; x_1,\ldots x_n\right]\right| < \varepsilon \text{ holds.}$$

The result contained in the following lemma is immediately and we omite the proof.

Lemma 2.3. Let $I \subset R$ be a compact real interval, $f \in C_b(I^n)$. Then f is uniform B-continuous on I^n.

Now, we are ready to establish the main result of the paper. It is contained in

Theorem 2.1. *Let $f \in C_b(I^n)$ be arbitrarily chosen. For any positive number ε there exist the positive numbers $A_i = A_i(\varepsilon)$ $\left(i = \overline{1,n}\right)$ so that the following inequality*

(2.10) $$\left|\Delta s_1,\ldots,s_n\left[f; x_1,\ldots,x_n\right]\right| \le \frac{\varepsilon}{n+1} + \sum_{i=1}^{n} A_i(\varepsilon)(x_i - s)^2$$

holds, for any points $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$.

Proof. Because $f \in C_b(I^n)$, from lemma 2.3 it follows that f is uniform B-continuous on I^n, i.e. $(\forall)\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ so that for any points $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ satisfying the conditions $|x_i - s_i| < \delta$ $\left(i = \overline{1,n}\right)$ the inequality

(2.11) $$\left|\Delta s_1,\ldots,s_n\left[f; x_1,\ldots,x_n\right]\right| \le \frac{\varepsilon}{n+1} \text{ holds.}$$

Because $f \in C_b(I^n)$ applying the lemma 2.2 f is B-bounded on I^n, i.e. there exists K>0 so that for any $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ the inequality

(2.12) $$\left|\Delta s_1,\ldots,s_n\left[f; x_1,\ldots,x_n\right]\right| \le K \text{ holds.}$$

Next, let us to separate the points $(x_1,\ldots,x_n),(s_1,\ldots s_n) \in I^n$ in the following classes:

-points with the property $|x_i - s_i| < \delta$, for all $i \in \{1,2,\ldots,n\}$;

-points with the property $|x_i - s_i| < \delta$ for $(n-1)$ values $i \in \{1,2,\ldots,n\}$;

-points with the property $|x_i - s_i| < \delta$ for a unique value $i \in \{1,2,...,n\}$;

-points in with $|x_i - s_i| < \delta$ for $(n-1)$ for all $i \in \{1,2,...,n\}$.

For the points from the first class, the inequality (2.11) holds.

Let us now to consider the points from the second class, i.e. the points for which there exists a unique $i \in \{1,2,...,n\}$, with $|x_i - s_i| < \delta$.

If $j = 1$, then $(x_1, x_2, ..., x_n),(s_1, s_2, ..., s_n)$ with the above property, one obtains:

(2.13) $$|\Delta s_1 s_2, ..., s_n[f; x_1, ..., x_n]| \le K \cdot \delta^{-2} \cdot (x_1 - s_1)^2.$$

Because the coordinate j for which $|x_j - s_j| < \delta$ can be arbitrarily chosen in the set $\{1,2,...,n\}$, it follows that for the pairs $(x_1, ..., x_n),(s_1, ... s_n) \in I^n$ for which exist one only $i \in \{1,2,...,n\}$ so that $|x_j - s_j| \ge \delta$, the inequality

(2.14) $$|\Delta s_1, ..., s_n[f; x_1, ..., x_n]| \le K \cdot \delta^{-2} \sum_{i=1}^{n} (x_i - s_i)^2$$ holds.

In a similar way, one deduces for the points $(x_1, ..., x_n),(s_1, ... s_n) \in I^n$ for which there exist only two values $i, j \in \{1,2,...,n\}$, $i \ne j$ so that $|x_i - s_i| \ge \delta$, $|x_j - s_j| \ge \delta$ the inequality

(2.15) $$|\Delta s_1, ..., s_n[f; x_1, ..., x_n]| \le K \cdot \delta^{-2^2} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} (x_i - s_i)^2 (x_j - s_j)^2$$

Continuing this reason, it follows that for the points $(x_1, ..., x_n),(s_1, ... s_n) \in I^n$ having the property $|x_i - s_i| \ge \delta$ for any $i \in \{1, ..., n\}$, the inequality

(2.16) $$|\Delta s_1 s_2, ..., s_n[f; x_1, ..., x_n]| \le K \cdot \delta^{-2^n} \cdot (x_1 - s_1)^2 ... (x_n - s_n)^2.$$

From all the above observations, we can conclude that for any $(x_1, ..., x_n),(s_1, ..., s_n) \in I^n$ the inequality

(2.17) $$|\Delta s_1, ..., s_n[f; x_1, ..., x_n]| \le K \cdot \delta^{-2^1} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} (x_i - s_i)^2 (x_j - s_j)^2 +$$

$$+ K \delta^{-2^2} \sum_{\substack{i,j=1 \\ i \ne j}} (x_i - s_i)^2 (x_j - s_j)^2 + ... + K \cdot \delta^{-2^n} (x_1 - s_1)^2 (x_2 - s_2)^2 ... (x_n - s_n)^2$$

holds.

Because $s_k, x_k \in [0,1]$, $(\forall) k = \overline{1,n} \Rightarrow |x_k - s_k| \le 1$, for any $k \in \{1,2,\ldots,n\}$. By using this last observation, from (2.17) it follows:

$$(2.18)\ \left|\Delta s_1,\ldots,s_n[f;x_1,\ldots,x_n]\right| \le \frac{\varepsilon}{n+1} + K \cdot \delta^{-2}\left\{1+\delta^{-2}+\ldots+\delta^{-2^n+2}\right\}.$$

$$(x_1 - s_1)^2 + K \cdot \delta^{-2^2}\left\{1+\delta^{-2}+\ldots+\delta^{-2^n+2^2}\right\}(x_2 - s_2)^2 + \ldots + K \cdot \delta^{-2}(x_n - s_n)^2$$

. Choosing now:

$$A_1(\varepsilon) = K \cdot \delta^{-2}\left\{1+\delta^{-2}+\delta^{-2^2}+\ldots+\delta^{-2^n+2}\right\}$$

$$A_2(\varepsilon) = K \cdot \delta^{-2}\left\{1+\delta^{-2}+\delta^{-2^2}+\ldots+\delta^{-2^n+2^2}\right\}$$

....

$$A_k(\varepsilon) = K \cdot \delta^{-2}\left\{1+\delta^{-2}+\delta^{-2^2}+\ldots+\delta^{-2^n+2^k}\right\}$$

....

$$A_n(\varepsilon) = K \cdot \delta^{-2}$$

from (2.18) it follows (2.7) and the proof ends.

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