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TOPOLOGICAL MANIFOLDS WITH A L_p STRUCTURE

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Abstract. In this paper we present the topological manifolds with L_p-structure. We describe the riemannian structure which determine the Hilbert-space, and prove some interesting properties. Finnaly it is presented an example for this manifolds

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Definition 1: A Riemmanian structure g on U is a measurable function of Lebesgue class and theirs values are in the set of R^n Euclidian structures. Such a structure determines a $\Omega^k(U,g)$ Hilbert space formed of ω complex, measurable differential forms of Lebesgue class, of k degree that verify the relation:

$\|\omega\|_g^2 = \int_U \lambda^k(g)(\omega,\omega)\mu_g < +\infty$, where $\lambda^k(g)$, (and μ_g) describes a quadric associated in a canonical way to g on $\Lambda_C^k(T^*U)$. We define $\tau : U \to L(\Lambda_C(R^{n,*}))$ *de Hodge* involution field determined by g: $\tau\omega = \begin{cases} i^{p(p-1)+m} * \omega, \text{ if } n = 2k \\ i^{p(p-1)+m+1} * \omega, \text{if } n = 2k-1 \end{cases}$ where p is the degree of ω, and * is the unitary field determined by $x \in U$.

Definition 2: Let *R (U)* is an ensemble of Riemmanian structures g on U, which exists for $k = m, m+1$, and $1 \le q_k \le 2 \le p_k < +\infty$, real, B > 1 satisfying the next relations:

(1) $L^{p_m}\left(U,\Lambda_C^k\left(T^*U\right)\right)\subset\Omega^k(U,g)\subset L^{q_m}\left(U,\Lambda_C^k\left(T^*U\right)\right)$

(2) $\frac{1}{p_m}+\frac{1}{n}>\frac{1}{q_m+1}$

(3) $g>B$

Lemma 3 : Let p, q and N >n real positive so that $p^{-1}+q^{-1}=1$ and $p^{-1}+N^{-1}=q^{-1}$ and let $n_o=nN(N-n)^{-1}$. Whatever $\xi\in L^q\left(V,\Lambda_C\left(T^*V\right)\right)$, $\delta_0(P_0+\Delta_0)^{-1}\xi$ belongs to $L^p\left(V,\Lambda_C\left(T^*V\right)\right)$. The defined application is continuous and belongs to $L^{n_o+}\left(L^q\left(V,\Lambda_C\left(T^*V\right)\right),L^p\left(V,\Lambda_C\left(T^*V\right)\right)\right)$. Let $\theta:U\to W$ an homeomorphism derivable on opened from R^n, whose derivate θ' belongs to $L^p\left(U,End\left(R^n\right)\right)$ for $p\geq 1$. We allege then that θ is derivable of p degree. Taking into consideration a theorem of Y. Reshetnyak in [9], an homeomorphism like that preserves the class of measure Lebesque, and its derivate operates on measurable sections of tangent fibrate.

Definition 4: A topological variety is called derivable of *p* degree if the map transformations associated to its atlas are derivable of *p* degree.

It is possible then to define Riemmanian structures on V and the Hilbert space of the differentiable forms measurable on V of *k* degree and of square incorporable on a g structure which will be written down with $\Omega^k(V,g)$. Let a covering $O=(O_i)_{i\in I}$ of the open maps $\theta:O_i\to U\subset R^n$ we will jot down with $R(O)$ the Riemannian structures g where g_i notes the images on U_i of the g restrictions at O_i with $g_i\in R(U_i)$ and we will put $n(g)=\sup_i n(g_i)$. Let $B(O)\subset C(V)$ the dense subalgebra generated by subalgebras $C_c^\infty(U_i)$pentru $i\in I$ i.e generated by the elements as $f=f_1+f_2+...+f_N$, unde $f_i\in C_c^\infty(U_i)$.

If $g \in R(O)$, taking into consideration the relation (3) from the definition no. 2 we will demonstrate that the subspace generated by the union $C_o\left(U_i, \Lambda_C^k\left(T^*U_i\right)\right)$ is a dense subspace of $\Omega^k(V, g)$. Thus the space $R(O)$ becomes a metrizable space for a family of semi- metrics determined by positive functions:

$g \mapsto \|\omega\|_g^2$; $g \mapsto p_k(g_i)$; $g \mapsto q_k(g_i)$ where ω traverse the union $C_o\left(U_i, \Lambda_C^k\left(T^*U_i\right)\right)$ and $p_k(g_i)$ (and $q_k(g_i)$), for $k=m$, $m+1$ is the smallest (respective the highest) real number for which the inclusions from the definition no. 2 are verified on U_i .

Proposition 5. The metrizable topological space $R(O)$ is connected by arks.

Proof : This result is a consequence of the complex interpolation. Let U an opened relatively compact from R^n, şi g_o, g_1 two elements from $R(U)$, $p_k^0, p_k^1, q_k^0, q_k^1$, associated real numbers, which verify the condition (1) from the definition no 2 . There is a continuous inclusion of $\Omega(U, g_i)$, $(i=0,1)$ in $L^p\left(U, \Lambda_C\left(T^*U\right)\right)$ with $q=\min(q_0, q_1)$ so we may apply the complex interpolation to the spaces couple Hilbert $\left(\Omega^*(U, g_0), \Omega(U, g_1)\right)$.

For $X \in R^n$, notam $< X, X >$ the standard Euclidian scalar product, and for $i=0$ and $i=1$ let $x \mapsto A_i(x)$ a measurable field on U of positive matrices for any $X \in R^n$, for which we have equality almost everywhere in Lebesgue measure, $g_i(x)(X, X) = < A_i(x)X, X >$.

Let $t \in [0,1]: A_t = A_0^{1/2}\left(A_0^{-1/2} A_1 A_0^{-1/2}\right)^t A_0^{1/2}$

Then we have a canonical identification of $\left(\Omega^k(U,g_0),\Omega^k(U,g_1)\right)_t$ cu $\Omega(U,g_t)$, where $g_t(x)(X,X) = < A_t(x)X,X >$. For $\omega \in C^\infty\left(U,\Lambda_C^k\left(R^{n,*}\right)\right)$ we have indeed for $t \in [0,1]$,

$$\|\omega\|_{g_t}^2 = \int_U < \lambda^k\left(A_t^{-1}(x)\right)\omega,\omega > \sqrt{\det(A_t(x))}dx.$$

When applying the method from ([3], cap 5) we have in a canonical mode for ω of k degree, that the norm on $\left(\Omega^*(U,g_0),\Omega^*(U,g_1)\right)_t$ is given by :

$\int_U < B_t(x)\omega(x),\omega(x) > dx$, where

$B_t = \lambda^k(A_0)^{-1/2}\left(\lambda^k(A_0)^{1/2}\lambda^k\left(A_1^{-1}\right)\lambda^K(A_0)^{1/2}\right)^t\lambda^k(A_0)^{-1/2}\det\left(A_0(x)^{1-t}A_1(x)^t\right)^{1/2}$, i.e

$B_t = \lambda^k\left(A_t^{-1}\right)\sqrt{\det(A_t(x))}$.

The relations from the definition no 2 are being satisfied for $g(t)$ with:

$$\frac{1}{p_k(t)} = \frac{t}{p_k^0} + \frac{1-t}{p_k^1}, \qquad \frac{1}{q_k(t)} = \frac{t}{q_k^0} + \frac{1-t}{q_k^1}$$

The application $t \mapsto g(t)$ is also continuous.

Obsevation 6 . The operator $A_t = A_0^{1/2}\left(A_0^{-1/2}A_1A_0^{-1/2}\right)^t A_0^{1/2}$, pt. $t \in [0,1]$ is a particular case of the notion of linear positive mediation operator, introduced by T. Audo and F Kubo [1].

Obsevation 7 . Let $f : Z \to R(O)$ a continuous application. On the field $\varepsilon = \left(\Omega(V,f(t))\right)_{t\in Z}$ there is a unique field structure generated continually by the sections ω, $\omega \in \sum_i C_0\left(U_i,\Lambda_c^k\left(T^*U_i\right)\right)$. So we have on $\mathcal{E}$ a structure of $C_0(Z)$- the canonical Hilbertian mode associated to f.

Also we fix a covering O of V, an element $g \in R(O)$, and let m the whole part of $\frac{n}{2}$. We will begin by defining:

$d:\Omega^k(V,g)\to\Omega^{k+1}(V,g)$ Let $\omega\in\Omega^k$ and ω_i the image in U_i of its restriction at O_i.

We will say that $\omega\in\mathrm{dom}d$ if $(\forall)i,\omega_i\in\mathrm{dom}\,\mathrm{d}_{U_i}\subset\Omega^{k+1}(U_i,g_i)$ and we define then $d\omega$ provided that $d\omega$ restriction at U_i is equal to $d_{U_i}\omega_i$. The next lemma demonstrates us that this definition makes sense and d is a dens domain.

Lemma 8. Let U and W two opens from R^n şi $\theta:U\to W$ a derivable homeomorphism of $p\geq m+1$ degree and $g_1\in R(U), g_2\in R(W)$ Riemannian structures so that $\theta^*(g_2)=g_1$ and $\alpha\in\Omega^k(W,_2)$, so that $d_{w^\alpha}\in\Omega^{k+1}$. The next properties are true for $k=m-1$, m, if m is even and for $k=m$, if m is odd.

1) Thus there is $\beta_n\in C_c^\infty(R^n,\Lambda^k(T^*R^{n,*}))$, so that if α_n is the restriction of β_n la W, then $\lim\alpha_n=\alpha$ and $\lim d_{w^{\alpha_n}}=d_{w^\alpha}$

2) $\theta^*(\alpha)\in\mathrm{dom}\,d_U\subset\Omega(U,g_1)$ and $d_U\theta^*(\alpha)=\theta^*(d_{w^\alpha})$

Proof : Supposing that n is odd. If α support is compact in W then 1) is being reduced to the tor case. The general case is being deduced from transpositions. To demonstrate 2), it is sufficient to presume that α is the restriction at W of $\beta\in C_c^\infty(R^n,\Lambda^k(T^*R^{n,*}))$, because d is closed. Forasmuch $\theta'\in L^p$ cu $p\geq m+1$, the differential forms $\theta^*(\alpha)$şi $\theta^*(d\alpha)$ are from L^1.

Also let $\theta_\varepsilon:U\to R^n$ smooth applications (C^∞) (not necessarily bijective) that converge uniform towards θ and whose derivates θ'_ε converge towards θ' in L^p. For $\xi\in C_c^\infty(U,\Lambda^m(T^*U))$, the following equality is true:

$$\int_U\theta^*(\alpha)\Lambda d\xi=\lim\int_U\theta_\varepsilon^*(\beta)\Lambda d\xi=\lim(-1)^{m-1}\int_U\theta_\varepsilon^*(d\beta)\Lambda\xi=(-1)^{m-1}\int_U\theta^*(d\alpha)\Lambda\xi.$$

For the next theorem we will consider the next conditions:

V- is a an oriented compact variety of n dimension, which allows a derivable structure of $m+1$ degree, so that for all finite coverings of open maps $\theta_i : O_i \to U_i$ the space $R(O)$ is non- empty. For this variety we will be able to construct an analytical operator (if not taking into consideration an omotopy).

Theorem 9. *With the previous hypothesis, let a coverage O of V and $g \in R(O)$.*

If n is odd, the operator $D = \tau d$ on $\Omega^m(V, g)$ is defined densely, closed, auto adjunct and has a resolvent on its support in $L^{n(g)+}$.

If n is even, the operator $D = d + d^*$ from $\Omega^m(V, g)$ in $\Omega^{m-1}(V, g) \oplus \Omega^{m+1}(V, g)$ is densely defined, closed, uncommutative vis a vis τ and has a resolvent in $L^{n(g)+}$. Whatever n parity, for all $f \in \beta(O)$, the commutator $[D, f]$ is defined densely and edged (limited). The D operator is unique if not taking into consideration an omotopy. The last part of the theorem, for example in the even case, if O_1 is on the other hand a coverage of the open maps of V and $g_1 \in R(O_1)$ there is also a continuous field ε of Hilbert spaces on $[0, 1]$, so that $\varepsilon_0 = \Omega^m(V, g)$, $\varepsilon_1 = \Omega^m(V, g_1)$ and a family of continuous autoadjunct operators D on ε, a resolvent in the ideal of compact operators of Hilbertian module, so that $D_0 = D$, $D_1 = D_1$, is a dense involutional subalgebra of $C(V \times [0, 1])$ so the commutative elements of D are edged.

In the even case, we will have an analogous description and we put $\Omega^m(V, g)$, as $\Omega^m(V, g) \oplus \Omega^{m-1}(V, g) \oplus \Omega^{m+1}(V, g)$.

Proof: Let $\ell_i \in B(O)$ a partition of the unit associated to the covering. The d domain contains the subspace from Ω^m generated by subspaces $C_c^\infty(U_i, \Lambda^m)$ which are dense.

Let $\omega \in dom\, d$ and $\omega_i = l_i\omega$. According to the previous lemma, there is $\alpha_k \in C_c^\infty(U_k, \Lambda^m)$ that converges towards ω_i so that $d\alpha_k$ converges towards $d\omega_i$: we will demonstrate that if n is odd, τd is autoadjunct and if n is even, the adjunct of d is $-\tau d\tau$. Supposing that n is even and we proved that *imd* is closed.

Let d_i the closing of d on the essential domain $C_c^\infty(O_i, \Lambda_0^m(T^*O_i))$ and ${}^t d_i$ their transposed on $\Omega^m(O_i)$. So we know that if d_i image is closed, by duality, also *im* ${}^t d_i$ is closed. There is $\omega_k \in dom\ {}^t d_i$ so that ${}^t d_i \xi_i \lim(d\omega|O_i)$. We consider $\xi = \sum_i \ell_i \xi_i$ and $\lim d\omega_k = d\xi$, therefore *imd* is closed, respective the inverse d^{-1} of d is continuous on its support. Let P the octagonal projector of $\Omega^m(V, g)$, on d support, and Q the octagonal projector of Ω^{m+1} on *imd*. For each i, let $\theta_i \to U_i \subset T^n$ an immersion and we fix an extension h_i on T^n and g_i on U_i. We will identify the differential forms with the support in O_i and those ones on T^n with the support in U_i. Let $t: \Omega^{m+1}(T^n, h_i) \to \Omega^m(T^n, h_i)$ a continuous operator defined by lemma 3 and ξ of $C_{\Box}^\infty$ class.

$$t'\xi = \delta_0(P_0 + \Delta_0)^{-1}\xi .$$

Let $\psi_i \equiv 1$ on the $\sup p(\ell_i)$ and with the support in U_i the limited interval, $[d, \psi_i]$. We note with d_i the exterior difference on $\Omega(T^n, h_i)$. Let $\ell_i : \Omega^{m+1}(V, g) \to \Omega^m(T^n, h_i)$ given by $\ell_i(\alpha) = d_i \psi_i d^{-1} Q\alpha$ and we presume $T_\alpha = \sum \ell_i t_i \ell_i(\alpha)$.

We calculate dT this way: $dT_{\alpha} = \sum_i \ell_i t_i \ell_i(\alpha) + \sum_i [d, \ell_i] t_i \ell_i(\alpha)$.

We have $dT\alpha = Q + bQ$, where $b \in L^{n(g)+}(\Omega^{m+1}, \Omega^{m})$. Also we have $PT - d^{-1} \in L^{n(g)+}$ and therefore $d^{-1} \in L^{n(g)+}$.

In the case n- even, we obtain by an analogue reasoning that d image is closed and d=d_0, and $d^* = -\tau d\tau$ so $d + d^*$ anticommutes with τ. We construct the operator $T^k \in L(\Omega^{k+1}, \Omega^k)$, $T^k\alpha = \sum \ell_i t_i^k \ell_i^k(\alpha)$, where t_i^k, ℓ_i^k are being constructed as above. We obtain that $dT^{m-1} + T^m d = 1 + a$, with $a \in L^{n(g)+}$, then $d^{-1}, d^{*-1} \in L^{n(g)+}$ and finally that D is a resolvent in $L^{n(g)+}$. This property is demonstrated in [9]. For the last stage, let O_2 a covering that contains at least O şi O_1 and let $h_2 \in R(O_2)$. It is sufficient to demonstrate for h_2.

Let $g(t)$ an optimal way in $R(O)$ with $g(0) = h_2$ and $g(1) = g$ obtained by a complex interpolation, as in the proposition 5. As τd is autoadjunct on the space generated by $C_c^\infty(U_i, \Lambda^m)$ we have the that the family of operators $D = (D_t)_{t\in[0,1]}$ operates on $C([0,1])$ module $(\Omega^*(V, g(t)))_{t\in[0,1]}$ is adjunct and it is the resolvent in the ideal of the compact operators of the module.

Finally, let W_j, for $1 \le j \le k$, opened of the covering O_2. Subalgebra $C(V \times [0,1])$ generated by the union of the subalgebras $C_c^\infty(U_i \times [0,1])$ and $C_c^\infty(W_j \times [0,1])$ is dense and composed by elements that commute in D.

Exemple 10: Derivable varieties of p degree.

Let V is a topological variety whose maps that define its atlases are being defined by derivable homeomorphisms $\theta : U \to W$ unde $\theta' \in L^p$. The conditions from the theorem 9 are being fulfilled if $p > \frac{n(n+1)}{2}$. Taking into consideration the previous notations, let $(O_i)_{i\in I}$, a covering and $(g_i)_{i\in I}$, the specific metric of this covering. Let $A : U_i \to L(R^n)$ a measurable field of positive matrices on g_i, in comparison with the standard structure, and a norm on $(\Omega^k(U_i, g_i))$, given by

$$\left\{\int_{U_i} < \lambda^k(A^{-1})\omega, \omega > \det(A)^{\frac{1}{2}} dx\right\}^{\frac{1}{2}}.$$

If $a \in M_n(R)$, is a positive matrix ≥ 1 and we always have

$$\lambda^k(a^{-2})\det(a) \leq \|a\|^{n-k} \quad \text{and } \lambda^k(a^2)\det(a)^{-1} \leq \|a\|^k .$$

Therefore $\lambda^k(A^{-1})\det(A)^{\frac{1}{2}} \in L^{\frac{p}{n-k}}$ and $\lambda^k(A)\det(A)^{-\frac{1}{2}} \in L^{\frac{p}{k}}$, which according to Hölder inequality, lead us to the inclusion $L^{p_k} \subset \Omega^k(V, g) \subset L^{q_k}$, where $p_k = \frac{2p}{p+k-n}$ and $q_k = \frac{2p}{p+k}$. If $n=2m+1$ then the relations from the definition 2 are being satisfied if $\frac{p-m-1}{2p} + \frac{1}{2n} > \frac{p+m+1}{2p}$, i.e if $p > m(m+1) = \frac{n(n+1)}{2}$. If $n=2m$ then the relations from the definition 2 are being satisfied $\frac{p-m}{2p} + \frac{1}{n} > \frac{p+m+1}{2p}$, i.e if $p > \frac{n(2m+1)}{2} = \frac{n(n+1)}{2}$

References.

[1] T. Ando, F. Kubo- Means of positive linear operators, Math. Annalen 246 (1979) n.3, p.205-224.

[2] M.F. Atiyah, V. K. Patodi, I.M. Singer- Spectral asymmetry and Riemmanian Geometry I, Math. Proc. Camb. Phil. Soc. 77 (1975) p. 43-69.

[3] J. Berg, J. Löfström- Intrpolation Spaces, an introduction, Springer-Verlag: Berlin Heildelberg New York (1987).

[4] A. Connes- Non Commutative Geometry, Odile Jacob, Paris.

[5] A. Connes, D. Sullivan, N. Teleman- Quasiconformal mappings, Operators in Hilbert space, and local formulae for characteristic classes, Topology 33 n.4 (1984) p. 683-691.

[6] F. W. Gehring- QuasiConformal Mapping, Acta Math 56 (1989).

[7] I.C. Gohberg, M.G. Krein- Introduction to the theory of linear nonselfadjoint operators, American Mathematical Society, Transl. Of Math. Monographs (1969).

[8] M. Hilsum- Fonctorialité en K-théorie bivariante pour les variété lipschitziennes, K-theory 3 (1989) p. 401-440.

[9] Y.G. Reshetnyak- Some Geometrical properties of functions and mappings with generalized derivatives, Sibirsck. Mat.Ž 7 (1966) 886-919.

[10] E. Stein- Singular integrals and differentiabitity of functions, Princeton mathematical series 30 Princeton University Press, Princeton NJ (1970).

[11] N. Teleman- The index of signature operator on Lipschitz manifolds, Pub. Math. I.H.E.S. 58 (1983) p. 39-78.

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