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THE HARDY SPACES OF SOME AVERAGING OPERATORS

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Abstract. In this paper we determine the Hardy spaces for averaging operators, and for the averaging integral operator (10) we obtain the Hardy spaces of $I^n \left(I^n = \underbrace{I \circ I \circ \ldots \circ I}_{n}\right)$.

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1. Introduction

The notion of the averaging operation has been introduced by S.S.Miller and P.T. Mocanu [4] and is a generalized complex-valued version of the First Mean-Value Theorem for Riemann Integrals. Recall that in the real case if f and h are continuous on [0,1], with $h(x) \geq 0$, then there exists $c \in (0,x)$ such that

$$\int_0^x f(t)h(t)\,dt = f(c)\int_0^x h(t)\,dt. \tag{1}$$

If we let $h(x) = g'(x)$ with $g(0) = 0$, then

$$\frac{1}{g(x)}\int_0^x f(t)\,g'(t)\,dt = f(c) \in f([0,1]) \tag{2}$$

for all $x \in [0,1]$.

In complex - value analog of (2) is

$$\frac{1}{g(z)}\int_0^z f(w)g'(w)dw \in f(U) \tag{3}$$

for all $z \in U$, where f and g are analytic on U and satisfy some simple conditions (U is the unit disk).

Let H be the set of functions that are analytic in U and let $H_0 = \{h \in H \mid h(0) = 0\}$. If G is a set then we denote the convex hull of G by $C \circ G$.

If we denote the integral in (3) by $I[f]$ then (3) can be rewritten as

$$I[f](U) \subset f(U). \tag{4}$$

This turn implies that

$$I[f](U) \subset C \circ f(U). \tag{5}$$

This generalized complex - valued version of Riemann's mean value theorem be analyzed here.

2. Preliminaries

Definition 1. If $K \subset H$ and if an operator $I : K \to H$ satisfies

$$I[f](U) \subset C \circ f(U) \text{ with } I[f](0) = f(0) \tag{6}$$

for all $f \in K$, then I is said to be averaging operator on K.

A simple characterization of such operations is given with support of subordonations.

Definition 2. If f, F analytic in U, then function f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

Lemma 1. ([4]). *Let $I : K \to H$ satisfy $I[f](0) = f(0)$. A necessary and sufficient condition for I to be an averaging operation on K is that for all convex h*

$$f \prec h \Rightarrow I[f] \prec h. \tag{7}$$

If I is an averaging operator then condition (7) is immediately. Next suppose that condition (7) holds. If $C \circ f(U) = \mathbb{C}$ then condition (6) holds.

If $C \circ f(U) \neq \mathbb{C}$ then there exists a convex function $h: U \to C \circ f(U)$ with $h(0) = f(0)$ and $h(U) = C \circ f(U)$.

Lemma 2. ([4]). *Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$, and let $g \in H_0$ with*

$$\operatorname{Re}\frac{\gamma z g'(z)}{g(z)} > 0 \text{ in } U.$$

If I is defined by

$$I[f](z) = \frac{\gamma}{g(z)^{\gamma}}\int_0^z f(t)g(t)^{\gamma-1}g'(t)\,dt \tag{8}$$

then I is an averaging operator on H.

If we set $\gamma = 1$ in (8) we obtain

$$I[f](z) = \frac{1}{g(z)}\int_0^z f(t)g'(t)\,dt \in C \circ f(U) \tag{9}$$

This is a generalized complex analog of the mean-value theorem of real analysis as referred in (2).

For $g(z) = z$ in (9) we obtain

$$I[f](z) = \frac{1}{z}\int_0^z f(t)\,dt \in C \circ f(U) \tag{10}$$

that is, the "average value" of lies in the convex hull of $f(U)$.

This operator is similar to the Alexander operator

$$I[f](z) = \int_0^z \frac{f(t)}{t}\,dt. \tag{11}$$

However, this operator is not an averaging operator. If we set $F(z) = I[f](z)$ and $f(z) = z + \frac{z^2}{2}$, then $F(z) = z + \frac{z^4}{4}$. Since $F(1) < f(1)$, $F(-1) < f(-1)$ and the functions f and F are convex, we have $F(U) \not\subset f(U) = C \circ F(U)$.

For f analytic in U and $z = re^{i\theta}$ we denote

$$M(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}}, \text{ for } 0 < p < \infty,$$

$$M(r,f)=\sup_{0\le\theta\le 2\pi}\left|f(re^{i\theta})\right|, \text{ for } p=\infty$$

A function is said to be of Hardy class H^{∞}, $0<p<\infty$, if $M(r,f)$ remains bounded as $r\to 1^{-}$. H^{∞} is the class of bounded analytic functions in the disk.

3. Main results

Theorem. 1. *If I is an averaging operator on $K\subset H$, then $I[f]\in H^{\lambda}$, for all $\lambda<1$ and for all $f\in K$.*

Proof. Let $I:K\to H$ be an averaging operator. From Lemma 1 we have that for all h convex function and $f\in K$ with $f\prec h\Rightarrow I[f]\prec h$. Applying the subordination theorem of Littlewood we obtain

$$M_p(r,I[f])\le M_p(r,h).$$

In [2] we determined the Hardy spaces for the class of convex functions and we obtain that $h\in H^{\lambda}$, $\lambda<1$. Hence $I[f]\in H^{\lambda}, \lambda<1$, for all $f\in K$.

Corollary. 2. *If $\gamma\in\mathbb{R}$, $g\in S^{*}$, where S^{*} is the class of starlike functions and $f\in H$ then for the averaging integral operator I defined by (8) we have $I[f]\in H^{\lambda}$, $\lambda<1$.*

Because $g\in S^{*}$ we obtain $\operatorname{Re}\dfrac{\gamma z g'(z)}{g(z)}>0$ and for $f\in H$, the integral operator defined by (8) is an averaging integral operator. Hence we obtain the result.

Theorem 3. *If $f\in H$ with $C\circ f(U)\ne k(U)$ where $k(x)=a+\dfrac{b}{1-ze^{i\tau}}$, $a,b\in\mathbb{C}$, $\tau\in\mathbb{R}$ (k is the function of Koebe), then for averaging integral operator I defined by (10) we have:*

(i) $\quad I[f](z)=\dfrac{1}{z}\displaystyle\int_0^z f(t)\,dt\in H^{1}$ *for all* $f\in H$

(ii) $I^n[f]\in H^\infty$, for all $n\in\mathbb{N}, n\geq 2$ and for all $f\in H, (I^n=\underbrace{I\circ I\circ\ldots\circ I}_{n})$.

Proof. i) From Theorem 1 we have that $f\in H^\lambda, \lambda<1$. Because $C\circ f(U)\neq k(U)$ we obtain $I[f]\prec g$, where g is a convex function and is not the function of Koebe. From the subordination theorem of Littlewood we have $M(r,I[f])\leq M(r,g)$, and $g\in H^1$ [2]. Hence $I[f]\in H^1$.

ii) If we denote with $F(z)=\int_0^z f(t)\,dt$ then $I[f]$ and F have the same class Hardy because:

$$\lim_{r\to 1^-} M(r,I[f])=\lim_{r\to 1^-}\left(\frac{1}{2\pi}\frac{1}{r}\int_0^{2\pi}\left|F(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}}=$$

$$=\lim_{r\to 1^-}\frac{1}{r}M(r,F)=\lim_{r\to 1^-}M(r,F).$$

On the other side if we denote $I[f](z)=\varphi(z)$ then from the integral theorem of Hardy-Littlewood we have:

$$\phi(z)=\int_0^z \varphi(t)\,dt\in H^\infty.$$

Hence we obtain that $I^2[f]\in H^\infty$.

Applying the method of induction theorem for $n\in\mathbb{N}, n\geq 2$ and using the integral theorem of Hardy-Littlewood we obtain the result.

Corollary 4. *If I is the averaging integral operator (10) and $f\in H$, with $C\circ f\neq k(U)$ where k is the function of Koebe, then $I^n[f]$ transforms all functions of H into the class of bounded functions in the unit disk, for all $n\in\mathbb{N}, n\geq 2$.*

Corollary 5. *If* $f \in H$ *and* $C \circ f \neq k(U)$, $C \circ f \neq \mathbb{C}$, *then for the integral operator of Libera:* $I[f](z) = \frac{2}{z}\int_0^z f(t)\,dt$ *we have:*

(i) $I[f] \in H^1$, *for all* $f \in H$;

(ii) $I^n[f] \in H^\infty$, *for all* $n \in \mathbb{N}, n \geq 2$ *and all* $f \in H$.

Because the integral operator Libera is $2I[f]$, where $I[f]$ is defined by (10) they have the same class Hardy.

Remark. In [3] we determined the Hardy classes for the Alexander's integral operator (11).

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