Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVII(2001), Nr.1-2, 65-70

THE NEWTON METHOD FOR INTEGRALl EQUATIONS OF HAMMERSTEIN TYPE

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Abstract. In this note we shall give an application of the Newton method concerning the approximation of the solutions of the integral equations of Hammerstein type. The particular form of this equation offers the possibility, as we shall see, to obtain relative simple convergence conditions for the Newton method. On the other hand, when the kernel of the integral operator is degenerated (or may be convenably approximated by such an operator), then the approximation of the solution reduces to the solving of a sequence of linear systems in R^n, though the setting of the problem is in an infinite dimensional space.

MSC:47H30

Keywords: Newton's method, convergence, degenerated kernel

1. The convergence of the Newton method

Consider the following integral equation of Hammerstein type:

$$\varphi(t) = g(t) + \lambda \int_a^b K(t,s) f(\varphi(s))\, ds, \tag{0.1}$$

where $g \in C([a,b])$, $k \in C([a,b] \times [a,b])$, $f \in C^2(R)$, $\lambda \in R$ and $a, b \in R$, $a < b$.

We attach to this equation the operator

$$P(\varphi)(t) = \varphi(t) - g(t) - \lambda \int_a^b K(t,s) f(\varphi(s))\, ds. \tag{0.2}$$

so (0.1) becomes

$$P(\varphi)(t) = 0 \tag{0.3}$$

The first and second order derivative of P are given by

(0.4) $$P'(\varphi)h(t) = h(t) - \lambda \int_a^b K(t,s) f'(\varphi(s)) h(s)\, ds,$$

(0.5) $$P''(\varphi)h(t)k(t) = -\lambda \int_a^b K(t,s) f''(\varphi(s)) h(s) k(s)\, ds,$$

where $h, k \in C([a,b])$ are arbitrary.

The Newton method for solving (0.5) consists in constructing the sequence of functions $(\varphi_n)_{n \geq 0}$, determined by the solving of a Fredholm integral equation at each iteration step :

(0.6) $$P'(\varphi_n)(\varphi_{n+1} - \varphi_n) + P(\varphi_n) = 0,\ \varphi_0 \in C([a,b]),\ n = 0, 1, \dots,$$

where φ_n is known and φ_{n+1} must be determined.

If we take into account (0.3), then (??) become

(0.7)

$$\varphi_{n+1}(t) = \lambda \int_a^b K_n(t,s) \varphi_{n+1}(s)\, ds + G_n(t);\ \varphi_0 \in C([a,b]);\ n = 0, 1, \dots,$$

where

(0.8) $$K_n(t,s) = K(t,s) f'(\varphi_n(s)),$$

and

(0.9) $$G_n(t) = g(t) + \lambda \int_a^b K(t,s) f(\varphi_n(s))\, ds - \lambda \int_a^b K_n(t,s) \varphi_n(s)\, ds.$$

Let $\varphi_0, u \in C([a,b])$ be two arbitrary functions. Consider the equation

(0.10) $$P'(\varphi_0) h(t) = u(t)$$

i.e.,

(0.11) $$h(t) - \lambda \int_a^b K(t,s) f'(\varphi_0(s)) h(s)\, ds - u(t) = 0.$$

Assume that the linear equation (0.11) has a unique solution, given by

$$h(t) = u(t) + \lambda \int_a^b K_0(t,s)\, u(s)\, ds \tag{0.12}$$

where $K_0(t,s)$ is the Fredholm solving kernel for (0.11).

Under the above assumption, with the notation from (0.10) it follows that there exists $[P'(\varphi_0)]^{-1}$ and, by (0.12) we get

(0.13)
$$\left\| [P'(\varphi_0)]^{-1} \right\| = \sup_{\|u\|=1} \left\| [P'(\varphi_0)]^{-1} u(t) \right\| \le 1 + \lambda (b-a) \sup_{1 \le t,s \le b} |K_0(t,s)| .$$

We shall use the following notations:

$$\alpha_0 = 1 + |\lambda| (b-a) \sup_{a \le t,s \le b} |K_0(t,s)| ; \tag{0.14}$$

$$\beta = \frac{|\lambda|}{2} (b-a) \sup_{a \le t,s \le b} |K(t,s)| \sup_{t \in R} |f''(t)| \tag{0.15}$$

$$B(\varphi_0, r) = \{ \varphi \in C([a,b]) \, \| \varphi - \varphi_0 \| \le r \} \tag{0.16}$$

where $r \in R$, $r > 0$, and $\|\cdot\|$ is the Chebyshev norm.

We may state the main result of this note:

Theorem 1 *If the functions $\varphi_0 \in C([a,b])$, $K \in C([a,b] \times [a,b])$, $f \in C^2(R)$ and the numbers α_0, β and r given by (0.14)-(0.16) verify*
i) equation (0.11) has a unique solution h given by (0.12);
ii) $p = \alpha_0 \beta r < 1$;
iii) $\delta_0 = \frac{\beta}{2} \delta^2 \| P(\varphi_0) \| < 1$, where $\delta = \frac{\alpha_0}{1-p}$;
iv) $\frac{2\delta_0}{\beta\delta(1-\delta_0)} \le r$,
then the following statements are true
j) the sequence $(\varphi_n)_{n \ge 0}$ is well defined;
jj) $\varphi_n \in B(\varphi_0, r), \forall n \ge 0$, and there exists the limit $\lim \varphi_n = \varphi^*$, $\varphi^* \in B(\varphi_0, r)$, the convergence being uniform;
jjj) the following inequalities hold:

$$\| \varphi^* - \varphi_n \| \le \frac{2\delta_0^{2^n}}{\beta\delta(1-\delta_0)}, \ n = 0, 1, \ldots .$$

P roof. t $\varphi \in B(\varphi_0, r)$ be an arbitrary function. From relation

$$\left\| [P'(\varphi_0)]^{-1} [P'(\varphi_0) - P'(\varphi)] \right\| \le \alpha_0 \beta \| \varphi - \varphi_0 \| \le \alpha_0 \beta r < 1,$$

using the Banach lemma, it results the existence of the application $[P'(\varphi)]^{-1}$ for all $\varphi \in B(\varphi_0, r)$ and moreover

$$\left\|[P'(\varphi)]^{-1}\right\| \leq \frac{\alpha_0}{1-p} = \delta, \tag{0.17}$$

Relations (??) and (0.17) imply

$$\|\varphi_1 - \varphi_0\| = \left\|[P'(\varphi_0)]^{-1} P(\varphi_0)\right\| \leq \alpha_0 \|P(\varphi_0)\| \leq \frac{\beta\delta^2}{\beta\delta}\|P(\varphi_0)\| \leq$$

$$\frac{2\delta_0}{\beta\delta(1-\delta_0)} \leq r,$$

i.e., $\varphi_1 \in B(\varphi_0, r)$.

Using the Taylor formula and relations (0.15) and (??) we get

$$\|P(\varphi_1)\| \leq \frac{\beta\alpha_0^2}{2}\|P(\varphi_0)\|^2 \leq \frac{\beta\delta^2}{2}\|P(\varphi_0)\|^2.$$

Denoting $\delta_1 = \frac{\beta\delta^2}{2}\|P(\varphi_1)\|$, then the above relation becomes

$$\delta_1 \leq \delta_0^2. \tag{0.18}$$

Suppose that for some $n \geq 1$, the elements $\varphi_1, \varphi_2, ..., \varphi_n \in B(\varphi_0, r)$ and, moreover

$$\|P(\varphi_i)\| \leq \frac{2\delta^{2^i}}{\beta\delta^2},\ i = \overline{0, n}.$$

We shall show that $\varphi_{n+1} \in B(\varphi_0, r)$ and $\left\|P(\varphi_{n+1})\right\| \leq \frac{2\delta_0^{2^{n+1}}}{\beta\delta^2}$.

Indeed,

$$\left\|\varphi_{n+1} - \varphi_0\right\| \leq_{i=0}^{n} \left\|\varphi_{i+1} - \varphi_i\right\| \leq_{i=0}^{n} \left\|[P'(\varphi_i)]^{-1}\right\| \|P(\varphi_i)\| \leq \delta_{i=0}^{n} \|P(\varphi_i)\|$$

$$\leq \delta_{i=0}^{n} \frac{2\delta_0^{2^i}}{\beta\delta^2} \leq \frac{2\delta_0}{\beta\delta(1-\delta_0)} \leq r$$

i.e., $\varphi_{n+1} \in B(\varphi_0, r)$. Next, as in the above reasoning,

$$\left\|P(\varphi_{n+1})\right\| \leq \frac{\beta\delta^2}{2}\|P(\varphi_n)\|^2 \leq \frac{\beta\delta^2}{2} \cdot \frac{\delta_0^{2^{n+1}} \cdot 4}{\beta^2\delta^4} \leq \frac{2\delta_0^{2^{n+1}}}{\beta\delta^2}.$$

It can be easily seen that in the Banach space $C([a,b])$ the sequence $(\varphi_n)_{n\geq 0}$ generated by the Newton method is fundamental and therefore convergent. Indeed, for all $m, n \in N$ we have

$$\left\|\varphi_{n+m} - \varphi_n\right\| \leq_{i=0}^{m-1} \left\|\varphi_{n+i+1} - \varphi_{n+i}\right\| \leq \delta_{i=0}^{m-1} \left\|P(\varphi_{n+i})\right\| \leq \frac{2\delta^{2n}}{\beta\delta(1-\delta_0)}.$$

Since $\delta_0 < 1$ it follows that $(\varphi_n)_{n\geq 0}$ is fundamental; denoting $\varphi^* = \lim \varphi_n$ we get jjj). ■

2. The case of the degenerated kernel

As we have seen, the Newton method requires the solving of a linear integral equation of Fredholm type at each iteration step. This problem simplifies if we assume that the kernel $K(t,s)$ from (0.1) is degenerated. In such a case, equations (0.7) will also have degenerated kernels and therefore their solving will reduce to the solving of some linear systems (in R^n).

Let $\alpha_i, \beta_i \in C([a,b])$, $i = \overline{1,p}$ where α_i, $i = \overline{1,p}$ and β_i, $i = \overline{1,p}$ are respectively linearly independent.

Suppose that the kernel $K(t,s)$ from (0.1) is of the form

$$K(t,s) = \sum_{i=1}^{p} \alpha_i(t)\,\beta_i(s). \tag{0.19}$$

In this case, equations (0.7) will have the following form

$$\varphi_{n+1}(t) = \lambda \sum_{i=1}^{p} \alpha_i(t) \int_a^b \beta_i(s)\, f'(\varphi_n(s))\, \varphi_{n+1}(s)\, ds + G_n(t). \tag{0.20}$$

We make the following notations:

$$x_i^{(n+1)} = \int_a^b \beta_i(s)\, f'(\varphi_n(s))\, \varphi_{n+1}(s)\, ds,\ i = 1,p;$$

$$a_{ij}^{(n+1)} = \int_a^b \alpha_i(t)\, \beta_j(t)\, f'(\varphi_n(t))\, dt,\ i,j = \overline{1,p};$$

$$\theta_j^{(n+1)} = \int_a^b G_n(t)\, \beta_j(t)\, f'(\varphi_n(t))\, dt,\ j = \overline{1,p}.$$

From (0.20) we deduce:

$$\varphi_{n+1}(t) = \lambda \sum_{i=1}^{p} x_i^{(n+1)} \alpha_i(t) + G_n(t),\ n = 0,1,\ldots. \tag{0.21}$$

The numbers $x_i^{(n+1)}$, $i = \overline{1,p}$ are determined from the system

$$X^{(n+1)} = \lambda A^{(n+1)} X^{(n+1)} + \theta^{(n+1)}$$

where

$$X^{(n+1)} = \left(x_1^{(n+1)}, x_2^{(n+1)}, \ldots, x_p^{(n+1)}\right)^T;$$

$$A^{(n+1)} = \left(a_{ji}^{(n+1)}\right)_{i,j=\overline{1,p}};$$

$$\theta^{(n+1)} = \left(\theta_1^{(n+1)}, \theta_2^{(n+1)}, \ldots, \theta_p^{(n+1)}\right)^T.$$

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Received: 11. 07. 2001

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