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ON A PROPERNESS METHOD FOR FREE VIBRATIONS PROBLEM

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Abstract. This paper is concerned with the existence of nontrivial periodic solutions (free vibrations) of semilinear wave equations of the form

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = f(t,x) \\ u(t,0) = u(t,\pi) = 0, & t \in \mathbf{R} \\ u(t+2\pi,x) = u(t,x), & t \in \Omega \end{cases}$$

where $\Omega = (0,2\pi) \times (0,\pi)$. It is used Liapunov-Schmidt method to obtain solutions of this problem as a limit of solutions of related problems in some finite dimensional spaces which can be solved. This problems will be considered with approximation schemes and corresponding mappings called approximation-proper operators.

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1. Introduction

Let X be a real Banach space and X^* the dual space of X, *i.e.* the Banach space of functionals on X. Denote by " $\rightarrow$ " and " $\rightharpoonup$ " the strong, respective the weak convergence on X.

Let $L : D(L) \subset X \rightarrow X^*$ be a linear operator, densely defined and let $S : X \rightarrow X^*$ be nonlinear. We are interested in approximative solving the semilinear equation

(1) $$Lu = S(u)$$

using the *Liapunov-Schmidt* method. We associate to equation (1.1) a sequence of related equations defined in finite dimensional subspaces of X^*, then the solution of (1.1) can be obtained as the limit of the sequence of the solutions of associated finite dimensional equations.

In fact, the approximation-solvability of the operator equation (1.1) is equivalent to its finite dimensional solvability whenever the given operators satisfy some properness conditions, as we can see next.

The theory of A-proper mappings combine pure existence results and constructive solvability of operator equations in a Banach space, via finite dimensional approximations.

Finally, we use these results to establish some existence theorems for free vibrations problem.

2. The Results

Assume that $N(L)$ is closed subspace of X, $\dim N(L) = \infty$ and $R(L) \subset X^*$ is also closed.

Let $P : X \to X$ be a *projector*, that is a linear, bounded, idempotent operator.

Lemma 1 *The operator $P^* : X^* \to X^*$ given by*

$$< P^* x^*, x > = < x^*, Px > \ , \ x^* \in X^* \, , \, x \in X$$

is projector.

Proof. The linearity and boundedness of P^* follow from the fact that P is linear and bounded. Then $< P^* P^* x^*, x > = < P^* x^*, Px > = < x^*, PPx > = < x^*, x >$, so P^* is idempotent and consequently, projector. ■

We say that the pair of operators (P, P^*) is *exact with respect to* L if

$$R(P) = N(L) \ , \ N(P^*) = R(L)$$

and

$$X = N(P) \oplus N(L) \ , \ X^* = R(L) \oplus R(P^*).$$

Lemma 2 *The restriction $L : D(L) \cap N(P) \to R(L)$ is invertible.*

Proof. Indeed, if $Lu = 0$, with $u \in D(L) \cap N(P)$, then $u \in N(L)$ and $u \in N(P)$. It follows $u = 0$ because $N(L) \cap N(P) = \{0\}$. ■

Now, let us denote

$$K := \left(L|_{D(L) \cap N(P)}\right)^{-1} (I - P^*),$$

where I is the identity on X^*. K is called the *generalized inverse* of L. Obviously, K is linear, and if L is closed, then K is continous, accordingly to the closed graph theorem.

Further, we use *Liapunov-Schmidt* method, therefore we apply P^* and $I - P^*$ in (1.1). We obtain the equivalent system $\begin{cases} P^* S(u) = 0 \\ L(I - P)u = (I - P^*)S(u) \end{cases}$, or $\begin{cases} P^* S(u) = 0 \\ u = Pu + KS(u) \end{cases}$, thus $u = T(u)$, where

$$T(u) := Pu + KS(u) - ZP^* S(u)$$

and $Z : R(P^*) \to N(L)$ is linear, continous and injective.

Now, let us consider an increasing sequence $(X_n)_{n\in\mathbf{N}}$ of finite dimensional subspaces of X and let $P_n : X \to X_n$ be projections such that $P_n x \to x$, $\forall x \in X$.

To each $n \in \mathbf{N}$ define $P_n^* : X^* \to X_n^*$, by the formula

$$< P_n^* x^*, x > = < x^*, P_n x >, \quad x^* \in X^*, x \in X_n.$$

It is easy to see that P_n^* are projections and moreover,

$$P_n^* x^* \to x^*, \quad \forall x^* \in X^*.$$

Under these assumptions, the scheme $\Gamma = \{X_n, P_n, X_n^*, P_n^*\}$ is a *complete projection scheme*, namely the following properties are satisfied:
(a) $\dim X_n = \dim X_n^*$.
(b) $P_n : X \to X_n$, $P_n^* : X^* \to X_n^*$ are projections such that $P_n x \to x$ and $P_n^* x^* \to x^*$, $\forall x \in X$, $x^* \in X^*$.

We give the following result for existence of existence of a complete projection scheme:

Lemma 3 *If* $\dim(L) = \infty$ *and the pair* (P, P^*) *is exact with respect to* L, *then there exists a complete projection scheme* Γ.

Proof. Let $\{U_n\}_{n\in\mathbf{N}}$ be an increasing sequence of finite dimensional subspaces of $N(L)$ such that $\overline{\bigcup_{n\in\mathbf{N}} U_n} = N(L)$ and $E_n : X \to U_n$ denotes the corresponding projections.

Now, define $X_n := U_n \oplus N(P)$ and the projections $\mathrm{P}_n := E_n + (I - P) : X \to X_n$. Similarly, we consider an increasing sequence $\{V_n\}_{n\in\mathbf{N}}$ of finite dimensional subspaces of $R(P^*)$ with $\dim V_n = \dim U_n$ and $\overline{\bigcup_{n\geq 1} V_n} = R(P^*)$. Let $F_n : X^* \to V_n$ be the corresponding projections and define $\mathrm{X}_n^* := V_n \oplus R(L)$, $P_n^* := F_n + (I - P^*) : X^* \to X_n$ In conclusion, $\Gamma = \{X_n, P_n, X_n^*, P_n^*\}$ is complete projection scheme for the pair (X, X^*). ■

In case $\dim N(L) = \infty$, we consider a kind of convergence defined in [7], [8] between the weak and strong convergence, called *L*-convergence.

We say that a sequence $\{u_n\}_{n\in\mathbf{N}} \subset X$ is *L-convergent* to $u \in X$ and denote $u_n \xrightarrow{L} u$ if

$$Pu_n \rightharpoonup Pu \quad \text{and} \quad (I - P)u_n \to (I - P)u.$$

Note that *I-convergence* is weak convergence and if $\dim N(L) < \infty$, then *L-convergence* is in fact strong convergence. To an equation

$$T(u) = f, \quad f \in X^*. \tag{2}$$

where $T : X \to X^*$, we associate a sequence of approximation-equations

$$T_n(u_n) = P_n^* f \,, \tag{3}$$

if $\Gamma = \{X_n, P_n, X_n^*, P_n^*\}$ is a complete projection scheme for the pair (X, X^*), and $T_n = P_n^* T$, $u_n \in X_n$.

Definition 4 *We say that the equation (2.1) is almost solvable with respect to* Γ *if there exists* $n_f \in \mathbf{N}$ *such that for all* $n \geq n_f$, *the equation (2.2) has a solution* $u_n \in X_n$, *and the sequence* $\{u_n\}_{n \geq n_f}$ *of approximative solutions is* L*-convergent to* $u \in X$ *and* $T(u) = f$.

Definition 5 *An operator* $T : X \to X^*$ *is called* A_L*-proper with respect to* Γ *if the restrictions* $T_n : X_n \to X_n^*$ *are continuous, for all* n, *and if* Γ_m *is a subscheme of* Γ *and* $\{u_m \mid u_m \in X_m\}$ *is a bounded sequence of solutions with* $T_m u_m \to f$ *in* X^* *it results* $u_m \xrightarrow{L} u$, *at least on a subsequence and* $T(u) = f$.

We give the following result concerning the almost solvability of a semilinear equation.

Theorem 6 *Let* X *be a real, reflexive Banach space. Assume that* K *is compact,* S *is bounded and the equation (2.1) is solvable in any finite dimensional subspace of* X *and the solutions remain in a bounded set* $\overline{\Omega} \subset X$. *Then if* $L - S$ *is* A_L*-proper, then the equation (2.1) is almost solvable.*

Proof. For $n \geq n_0$ the approximation equation

$$L_n u_n = S_n(u_n) \tag{4}$$

has a solution $u_n \in D(L) \cap \overline{X_n \cap \Omega}$, because (2.1) is finite dimensional solvable. We can suppose that $u_n \rightharpoonup u \in D(L) \cap \overline{X_n \cap \Omega}$ because X is reflexive. Let us consider the equivalent form of (2.1): $\begin{cases} P_n^* S(u_n) = 0 \\ (I - P)u_n = KS(u_n) \end{cases}$.

Because K is compact, $(I-P)u_n \to (I-P)u$, eventually on a subsequence. But $Pu_n \rightharpoonup Pu$, therefore $u_n L \to u$.

In conclusion, $(L - S)u_n \to 0$ and from the fact that the operator $L - S$ is A_L-proper, it follows: $Lu = S(u)$. ■

3. Application

Let us study now the existence of nontrivial periodic solutions (free vibrations) of semilinear wave equations of the form

$$\begin{cases} u_{tt} - u_{xx} + g(t, x, u) = f(t, x) \\ u(t, 0) = u(t, \pi) = 0, & t \in \mathbf{R} \\ u(t + 2\pi, x) = u(t, x), & t \in \Omega \end{cases} , \tag{5}$$

where $\Omega = (0, 2\pi) \times (0, \pi)$. Denote by $\tilde{C}^2$ the space of functions $v(t, x)$ of class C^2, 2π-periodic in t such that $v(t, 0) = v(t, \pi) = 0$.

In $L^2(\Omega)$, the family $(\psi_{nk})_{(n,k)\in \mathbf{N}\times\mathbf{K}}$, given by

$$\psi_{nk}(x, t) = \begin{cases} \sqrt{\frac{2}{\pi}} \sin nx \sin kt & (n, t) \in \mathbf{N} \times \mathbf{N} \\ \frac{1}{\pi} \sin nx & n \in \mathbf{N}\,,\ k = 0 \\ \sqrt{\frac{2}{\pi}} \sin nx \cos kt & n \in \mathbf{N}\,,\ -k \in \mathbf{N} \end{cases}$$

is an othonormal basis,

$$\Box\psi_{nk} = (n^2 - k^2)\psi_{nk}.$$

Define $L : D(L) \to L^2(\Omega)$, by

$$Lu = \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} (n^2 - k^2)(u, \psi_{nk})\psi_{nk}\,,$$

where

$$D(L) = \left\{ u \in L^2(\Omega) \mid \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} (n^2 - k^2) \left|(u, \psi_{nk})\right|^2 < \infty \right\}.$$

L is densely defined, selfadjoint and $R(L)$ is closed.

The generalized inverse K of L is compact. Indeed,

$$Ku = \sum_{k\neq n} \frac{(u, \psi_{nk})}{n^2 - k^2}\psi_{nk}.$$

Assume that $g : \Omega \to \mathbf{R}$ is a *Caratheodory* function and there exist $c > 0$, $g_0 \in L^2(\Omega)$ such that

$$|g(t, x, u)| \leq c\,|u| + g_0(t, x)$$

for all $(t, x) \in \Omega$ and $u \in \mathbf{R}$. Under these assumptions, the corresponding *Nemitki* operator of g, $S : L^2(\Omega) \to L^2(\Omega)$, given by

$$(Su)(t, x) = g(t, x, u(t, x))$$

is bounded and continous. We say that $u \in L^2(\Omega)$ is a generalized solution of (3.1) if

(6) $$(u, v_{tt} - v_{xx}) + (Su, v) = (f, v)\,, \quad \forall v \in \tilde{C}^2$$

Now, the equation (3.2) can be equivalently written as

$$Lu + S(u) = f\,, \quad u \in L^2(\Omega)$$

and we can apply theorem 1. In conclusion, equation (3.2) and consequently (3.1) have solutions in generalized sense.

References

[1] H. Amman, *Liusternik-Schnilermann Theory and Nonlinear Eigenvalue Problems*, Math. Ann., 119(1972), 55-72.

[2] V. Barbu, *Weak Solutions for Nonlinear Functional Equations in Banach Spaces*, Math. Ann, Pura Appl., 87(1970), 87-110.

[3] F.E. Browder, *Nonlinear Eigenvalue Problems and Galerkin Approximations*, Bull. AMS, 74(1968), 651-656.

[4] M.A. Krasnoselski, P.P. Zabreiko, *Geometric Methods in Nonlinear Analysis*, Springer Verlag, New York, 1984.

[5] L. Ljusternik, L. Schnilermann, *Methodes Topologique dans les Problemes Variationelles*, Hermann, Paris, 1934.

[6] C. Mortici, *A Topological Degree for A^*-proper Mappings Acting from a Banach Space into its Dual*, Bul. St. Univ. Baia Mare, 15(1999), nr.1-2, 139-144.

[7] D. Pascali, *On Mappings of Type (S)*, An. Univ. Ovidius, 3(1995).

[8] D. Pascali, *Elliptic Eigenvalue Variational Inequalities*, Lib. Math., 12(1992), 39-56.

[9] I.V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Transl. Math. Monographs, 139(1994).

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