Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVII(2001), Nr. 1-2, 77-84

SUMMABILITY OF DOUBLE FOURIER SERIES BY ROGOSINSKI-TYPE MEANS

Yuri NOSENKO

Abstract. Regularity conditions for Rogosinski-type (Bernstein-Rogosinski-type) means of double Fourier series of continuous functions are obtained. These means were first introduced in the most general form by R.M. Trigub. Such means depend on some numerical parameters, an appropriate borelian measure and its support and a shape of a domain generating partial sums of Fourier series. We investigate the case when the partial sums are generated by polygons (a rectangle, a rhombus,symmetric with respect to an origin) and the measure is uniformly distributed over the areas of correspondent polygons.

MSC: 42B08

Keywords: double Fourier series, linear means of Fourier series, regularity, Bern-stein-Rogosinski-type means, Rogosinski-type means.

1. Introduction

Let $T^2 = (-\pi, \pi]^2$, a function $f \in C(T^2)$ be 2π-periodic with respect to every variable,

$$f(x_1, x_2) \sim \sum_{(k_1,k_2)} c_{k_1,k_2} e^{i(k_1x_1+k_2x_2)} \tag{1}$$

be the Fourier series for f, c_{k_1,k_2} be the Fourier coefficients for f in the trigonometric system $\{ e^{i(k_1x_1+k_2x_2)} \}$.

Let W_0 be some domain from R^2 containing the origin inside itself, pW_0 be a homothetic transform of W_0 with the coefficient of homothety $p \in R$, i.e.

$$pW_0 = \{ (x_1, x_2) : (\frac{1}{p}x_1, \frac{1}{p}x_2) \in W_0 \} .$$

Then for $n \in N$

$$S_n(f; W_0, x_1, x_2) = \sum_{(k_1,k_2)\in nW_0} c_{k_1,k_2} e^{i(k_1x_1+k_2x_2)}$$

are the partial sums of (1) corresponding to W_0 (or generated by W_0).

The means

$$R_n(f; x_1, x_2) = \int_{R^2} S_n(f; W_0; x_1 - \frac{\gamma u}{n}, x_2 - \frac{\gamma v}{n}) d\mu(u, v) \tag{2}$$

were first introduced in this most general form and were investigated in different directions by R.M. Trigub [6]. These means are known as Rogosinski-type (Bernstein-Rogosinski-type) means. Here $n \in N, \gamma \in R, \mu$ is finite and normalized Borelian measure on R^2. Generally speaking, the means (2) depend on a numerical parameter γ, a choice of a measure μ and on a shape of a domain W_0.

In a discrete case when a measure α_k ($\sum_k \alpha_k = 1$ taking in account a condition of normalization of the measure) is concentrated at the points (u_k, v_k) the means (2) can be reduced to

$$R_n(f; x_1, x_2) = \sum_k S_n(f; W_0; x_1 - \frac{\gamma u_k}{n}, x_2 - \frac{\gamma v_k}{n}). \tag{3}$$

The partial sums of the series (1) correspondent to W_0 can be obtained from (2) or (3), say, by putting $\gamma = 0$ or concentrating the measure at the origin. In our results $\gamma \neq 0$.

We state here and investigate the problem of regularity of Rogosinski-type means, i.e. convergence of such means to a generating function. In our case it is a convergence of $R_n(f; x_1, x_2)$ to $f(x_1, x_2)$ itself in the uniform metric.

Two special classical cases of the means (2) were first introduced and investigated by W. Rogosinski and S. Bernstein in a case of homogeneously distributed measure at two points (see [6] for references) in one-dimensional situation (a measure 1/2 is concentrated at every point, these points are, of course, different for classical Rogosinski means and for Bernstein means).

Regularity and approximation properties of these classical means and some interesting generalizations in both one-dimensional and multiple cases were investigated by different authors (see for example [3] for references; another interesting generalization was proposed by A. Kivinukk in [1]). The means (2) and (3) were investigated in these directions as well. Regularity conditions of $R_n(f; x_1, x_2)$ were obtained in different cases of homogeneous and nonhomogeneous, discrete and continuous distributions of a measure (see [3], [5], [6] for results and references).

The means (2) can be represented in the equivalent form [6]

$$R_n(f; x_1, x_2) = \sum_{(k_1,k_2) \in Z^2} \phi(\frac{k_1}{n}, \frac{k_2}{n}) c_{k_1,k_2} e^{i(k_1 x_1 + k_2 x_2)}, \tag{4}$$

where Z^2 is an integer-valued lattice in R^2 and

$$\phi(x_1, x_2) = \chi_{W_0}(x_1, x_2) \int_{R^2} e^{-i\gamma(x_1 u + x_2 v)} d\mu(u, v). \tag{5}$$

Here χ_{W_0} is a characteristic function (indicator) of W_0.

The necessary condition of regularity of (2) in $C(T^2)$ for measurable in Jordan sense domain W_0 is (∂W_0 is the boundary of W_0)

$$I = \int_{R^2} e^{-i\gamma(x_1 u + x_2 v)} d\mu(u,v) = 0 \qquad (6)$$

for every point $(x_1, x_2) \in \partial W_0$ (see [6]).

Conditions of regularity of (2) in $C(T^2)$ can be divided in three main groups:

1) regularity of (3) for a discrete distribution of a measure μ both homogeneous or nonhomogeneous,

2) regularity of (2) for uniform (continuous) distribution of a measure μ along perimeters of different curves in R^2,

3) regularity of (2) for uniform (continuous) distribution of a measure μ along areas of different figures in R^2.

The problems of the first group are practically solved as conditions of regularity (both positive and negative) for (3) are obtained for arbitrary but right polygons centered at an origin as W_0 and for practically arbitrary polygons (positive only) as W_0 in [2], [4].

But a situation remains different for groups 2) and 3) as there are only separate isolated results for these groups (say, W_0 as a unit circle or as a unit square in [6], see also [4] for other results and references).

2.Main Results and Commentaries

We give here some positive results on regularity of Bernstein-Rogosinski-type means (2) in a case of homogeneous distribution of the measure μ over areas of some polygons. Here measure supports W^* and figures W_0 generating partial sums of (1) are both polygons (a rectangle, a rhombus). These results (theorems 1 and 2) give further extension and generalization of appropriate theorems 1 and 2 from [5].

These results including therem 3 are unified with the main idea – idea of "perpendicularity" in a certain sense of W_0 and W^*. This idea first appeared in a discrete case for W_0 as a parallelogram in [2]). We formulate this result here.

Theorem A. Let W_0 be a parallelogram which is symmetric with respect to an origin. The means (3) are regular in a case of a measure support consisting not less then of four points. In the case of four points the means (3) are regular if and only if these points are vertices of new parallelogram W^*. Morover sides of W_0 and W^* are mutually perpendicular, a measure μ is uniformly distributed at these points and a ratio of a product for lengths of two nonparallel sides of W_0 and W^* is equal to a ratio of two odd numbers.

This idea of perpendicularity for W_0 and W^* works in the case of regularity of the means (2). We demonstrate it in our new results. Now we will first state these results (theorems 1-3).

Theorem 1. Let a function $f \in C(T^2)$ be 2π-periodical with respect to every variable with the Fourier series (1). Let W_0 be a rectangle with sides $2a$ and $2b$ parallel to coordinate axes and symmetric with respect to an origin. Then the means (2) are regular only if a measure μ is uniformly distributed along an area of a new rectangle W^* with sides $2c$ and $2d$ which are parallel to coordinate axes. In this case a ratio of a product for two mutually perpendicular sides of W_0 and W^* is equal to a ratio of two natural numbers and $\gamma = \frac{\pi n}{ac}$ for $n \in N$.

Theorem 2. Let the function f be as in theorem 1. Let W_0 be a rhombus with diagonals $2a$ and $2b$ and with vertices on coordinate axes. The means (2) are regular only if the measure μ is uniformly distributed over an area of a new rhombus W^* which is like to the rhombus W_0 with the likeness coefficient p, is rotated by $\frac{\pi}{2}$ with respect to an origin, and $\gamma = \frac{2\pi k}{pab}$ for $n \in N$.

R.M. Trigub in [6] has proved that for $f \in C(T^2)$ with partial sums of (1) formed with the help of a unit square (W_0) and a support of a measure μ as a perimeter of W_0 (∂W_0) the means (3) are irregular. One can extend this result very easy from a unit square to a rectangle which is symmetric with respect to an origin with sides parallel to coordinate axes (the means (3) remain irregular).

Taking in account these considerations and our theorem A we come to the conclusion. Let W_0 be a symmetric with respect to an origin rectangle with sides parallel to coordinate axes and the support of the measure μ be another rectangle W^* then Bernstein-Rogosinski-type means (2) or (3)

1) can be regular if the measure is homogeneously concentrated at the vertices of W^* (theorem A),

2) are irregular if the support of the homogeneous measure is a perimeter of W^* [5],

3) can be regular once again if the homogeneous measure is concentrated over an area of W^* (theorem 1).

A problem arises. How one can reduce the measure support from an area of W_0 in a direction of its perimeter the means (2) to remain regular? Or how one can extend a measure support from a perimeter of W^* in a direction of an area of W_0 the means (2) to became regular?

The following theorem is valid in this direction.

Theorem 3. Let a function f be as in theorem 1. Let W_0 be a rectangle with sides $2a$ and $2b$, W^* be a rectangle with sides $2c$ and $2d$ and at last W^{**} be a rectangle with sides $2(c-\epsilon)$ and $2(d-\epsilon)$. The sides of every rectangle are parallel to coordinate axes and every rectangle is symmetric with respect to an origin. Let further the measure μ is uniformly distributed over the area of a figure $W^* \setminus W^{**}$ (a union of corresponding strips). Then the means (2) corresponding to these f, W_0 and μ are regular if a ratio for products of mutually perpendicular sides of W_0 and W^* is rational,

$$\gamma = \frac{\pi n}{bd} \; (\gamma = \frac{\pi k}{ac})$$

for $n \in N$, $k \in N$ and at last ϵ is rational multiple to c or d.

It is easy to notice that one can take different values for ϵ, i.e. theorem 3 remains valid for different values of ϵ along different coordinate axes (say, ϵ_1 and ϵ_2). These values of ϵ are not arbitrary in this case, of course.

3.Auxiliary Results

Let us take a matrix $\Lambda = \| \lambda_{k_1,k_2}^{(m,n)} \|$ with elements depending on $m \in N, n \in N$. Let us form now the linear means for (1) using this matrix, namely,

$$\tau_{m,n}(f;\Lambda) = \tau_{m,n}(f;\Lambda, x_1, x_2) \sim \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \lambda_{k_1,k_2}^{(m,n)} c_{k_1,k_2} e^{i(k_1x_1+k_2x_2)} \tag{7}$$

. It is well known these means are regular if $lim_{m\to\infty,n\to\infty} \lambda_{k_1,k_2}^{(m,n)} = 1$ and corresponding Lebesgue constants (norms of operators (7)) are bounded in m and n uniformly.

To check regularity of the means (7)(the means (2) or (3) are the special cases of (7), see (4)) we will use two results due to R.M. Trigub [6] and V.P. Zastavnyi [7]. They are formulated here as lemmas 1 and 2 respectively.

Lemma 1 (Trigub R.M.). If the elements of the matrix Λ are the values of the finite function $\psi(u,v)$ for $u = \frac{k_1}{m}, v = \frac{k_2}{n}$ with the support $[-1,1]^2$ and: 1) $\phi \in Lip\ (\frac{1}{2}+\epsilon), \epsilon > 0$ in C in u (uniformly in v); 2) the same in v (uniformly in u); 3) ϕ as a function of u has bounded in v number of points of inflection; 4) the same for ϕ as the function on v, then the Lebesgue constants, corresponding to Λ, are bounded in m and n.

Lemma 2 (Zastavnyi V.P.). If μ is a finite complex Borelian measure in R^2 with a complex support then for every $a > 0$ functions $Re\ g(x,y)$ and $Im\ g(x,y)$ where

$$g(x,y) = \int_{R^2} e^{-i(ux+vy)} d\mu(u,v),$$

taken over $[-a,a]^2$ as the functions of x and y have bounded (with respect to other variable) number of zeroes.

4. Proofs of Main Results

To prove theorems 1-3 we will first use the condition (6) and find W_0 and γ satisfying this condition and only then we check the boundedness of corresponding Lebesgue constants for obtained shapes of W_0 and values of γ using lemmas 1, 2.

Proof of theorem 1.

Let f, W_0, μ, W^* be as in theorem 1. Evaluation of the integral in (6) (we will omit some technical detailes) in our case gives us

$$I = (cd\gamma^2 x_1 x_2)^{-1} sin(\gamma x_1 c) sin(\gamma x_2 d).$$

The condition (6) applied for the boundary ∂W_0 of W_0, namely for the part with the equation $x_2 = |b|$ gives us that $sin(\gamma bd) = 0$. Arguing in the same way for other part of ∂W_0 we have $sin(\gamma ac) = 0$. Using these relations we obtain for $n \in N, k \in N$

$$\frac{ac}{bd} = \frac{n}{k}, \gamma = \frac{\pi n}{ac}.$$

Now we will prove regularity of the means (2) for our case. In accordance with (4), (5), (7) the elements of the matrix Λ which defines our linear means are values of the function (here χ_{W_0} is a characteristic function of W_0)

$$\phi(x_1, x_2) = \chi_{W_0} (cd\gamma^2 x_1 x_2)^{-1} sin(\gamma x_1 c) sin(\gamma x_2 d), \ (x_1 x_2) \neq 0,$$

$$\phi(0, x_2) = \chi_{W_0} (\gamma d x_2)^{-1} sin(\gamma d x_2) \ (x_2 \neq 0),$$

$$\phi(x_1, 0) = \chi_{W_0} (\gamma c x_1)^{-1} sin(\gamma c x_1) \ (x_1 \neq 0),$$

$$\phi(0,0) = 1.$$

Taking in account the value of ϕ at $(0,0)$ we notice that it remains to prove only boundedness of Lebesgue constants corresponding to $\phi(x_1, x_2)$ to complete the prooof of the theorem. Let us return to lemma 1. As $|\phi'_{x_1}(x_1, x_2)| \leq C_1$ and $|\phi'_{x_2}(x_1, x_2)| \leq C_2$ for $(x_1, x_2) \in W_0 \, \partial W_0$ then $\phi \in Lip\,1$ in x_1 for every x_2 and in x_2 for every x_1. Here C_1 and C_2 are absolutely constants. This means that the conditions 1) and 2) of lemma 1 hold with $\epsilon = \frac{1}{2}$. The conditions 3) and 4) of the same lemma are practically evident. But to prove them one has to use lemma 2 for the function $g(x_1, x_2) = f''_{x_1 x_1}(x_1, x_2)$ and $g(x_1, x_2) = f''_{x_2 x_2}(x_1, x_2)$. It remains us now to use lemma 1.

To prove the boundedness of the correspondent Lebesgue constants one can use another approach based on conditions of integrability of trigonometric series as it is done in [2].

Proof of theorem 2.

Let f, W_0, μ, W^* be as it is formulated. In this case W^* has diagonals pb and pa corresponding to diagonals of W_0. Evaluation of the integral in (6) for these data gives us (we will omit some technical detailes) for $x_1 b \neq \pm x_2 a$

$$I = C(1 - e^{i\gamma p(x_1 b + x_2 a)})(1 - e^{i\gamma(-x_1 b + x_2 a)}).$$

$$e^{-i\gamma pax_2}(x_1^2b^2 - x_2^2a^2)^{-1}$$

(constants C here and below are independent of x_1 and x_2).

As $I = 0$ at the boundary ∂W_0 of W_0 in accordance with (6)) then for $x_1b + x_2a = ab$ (an equation of a part of the boundary) the relation (6) is possible if $e^{i\gamma pab} = 1$. This relation gives us $\gamma = \frac{2\pi k}{pab}$ for $k \in N$. The same situation will be for other parts of ∂W_0.

Elements of the matrix which defines corresponding linear means (7) or (4) in our case are the values of the function

$$\phi(x_1, x_2) = C\chi_{W_0}(1 - e^{i\gamma p(x_1b+x_2a)})(1 - e^{i\gamma(-x_1b+x_2a)}).$$

$$e^{-i\gamma pax_2}(x_1^2b^2 - x_2^2a^2)^{-1}.$$

At the points of indeterminacy the function $\phi(x_1, x_2)$ has to be defined by continuity. The same technique (lemmas 1 and 2) has to be used to complete the proof.

Proof of theorem 3.

Let f, W_0, μ, W^* be as it is formulated. Evaluation of the integral in (6) for these data gives us (C is independent of x_1 and x_2 and $x_1x_2 \neq 0$)

$$I = Cx_1^{-1}x_2^{-1}(sin(\gamma x_2(d - \epsilon))sin(\gamma x_1(c - \epsilon)) - sin(\gamma x_1c)sin(\gamma x_2d))$$

and I is defined by continuity for $x_1x_2 = 0$. To find γ we will use (6) now. The equations of ∂W_0 are $|x_1| = a$, $|x_2| = b$. For these equations we have the systems

$$\begin{cases} sin(\gamma a(c - \epsilon)) = 0, \\ sin(\gamma ac) = 0 \end{cases}$$

and

$$\begin{cases} sin(\gamma b(d - \epsilon)) = 0, \\ sin(\gamma bd) = 0. \end{cases}$$

respectively. These systems give us

$$\gamma ac = \pi k, \gamma a\epsilon = \pi l, \gamma bd = \pi m, \gamma b\epsilon = \pi n$$

for $k, l, m, n \in N$,

$$\frac{bd}{ac} = \frac{m}{n},\ \epsilon = \frac{lc}{k}\ or\ \epsilon = \frac{nd}{m},$$

and at last

$$\gamma = \frac{\pi n}{bd}\ or\ \gamma = \frac{\pi k}{ac}.$$

Elements of a matrix defining correspondent linear means (7) or (4) in our case are the values of the function ($x_1x_2 \neq 0$)

$$\phi(x_1, x_2) = C\chi x_1^{-1}x_2^{-1}(sin\gamma x_1(c - \epsilon))sin(\gamma x_2(d - \epsilon)) -$$

$$sin(\gamma x_1 c) sin(\gamma x_2 d).$$

for $x_1 x_2 \neq 0$ and for $x_1 x_2 = 0$ the definition of the function ϕ has to be completed by continuity.

The same technique with lemmas 1 and 2 has to be used to complete the proof of this theorem.

References

1. **A. Kivinukk,** Subordination in Fourier approximation by generalized Rogosinski means and its applications, Bolyai Society Mathem. Studies, 5, (1996), 237–245.

2. **Yu. L. Nosenko,** Summation of double Fourier series by methods of Bern- stein-Rogosinski type, Theory of Mappings and Approximation of Functions, "Naukova Dumka", Kiev, 1983, pp. 89–97 (in Russian).

3. **Yu. L. Nosenko,** Regularity and approximation properties of means of Bernstein-Rogosinski type of double Fourier series, Ukrain. M. Zh., 37 , (1985), no. 5, 599–604 (in Russian); English transl. in Ukr. math. J., 37, (1985), no. 5, 485–490.

4. **Yu. L. Nosenko,** Regularity of Bernstein-Rogosinski-type means , Theory of Approx. of funct. Proc. of the Inst. of applied math. and mech., Donetsk, (1998), no. 3, 182–186 (in Russian).

5. **Yuri Nosenko,** Regularity conditions for Bernstein-Rogo- sinski-type means of double Fourier series, Buletinul Stiintific Univ. , Baia Mare (Romania), Ser. B, 16, (2000), no. 1, 93–98.

6. **R.M. Trigub,** Absolute convergence of Fourier integrals, summability of Fourier series, and polynomial approximation of functions on the torus, Izv. AN SSSR, Ser mathem., 44, (1980), no. 6, 1378–1409 (in Russian); English transl. in Math. USSR Izvestija, 17, (1981), no. 3, 567–593.

7. **V.P. Zastavnyi,** Concerning a set of zeroes for Fourier transform of a measure and summability of double Fourier series by Bernstein-Rogosinski type methods, Ukrain. M. Zh., 36, (1984), no. 5, 615–621 (in Russian).

Received 14.10.2001

Donetsk National Technical University,
Artema str., 58,
Donetsk, 83000,
UKRAINE
e-mail: yuri@nosenko.dgtu.donetsk.ua