Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVII(2001), Nr. 1-2, 85-92

A HALLEY-AITKEN TYPE METHOD FOR APPROXIMATING THE SOLUTIONS OF SCALAR EQUATIONS

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Abstract. The paper is concerned with approximation of the solutions of scalar equation by an iterative method of Halley-Aitken type. The local convergence and error bounds are discussed

MSC: 65H05, 65D15

Keywords: Halley iterative method, Newton iterative method, Aitken iterative method, convergence

1.Introduction

Let $f : [a, b] \to R$, where $a, b \in R$, $a < b$, and suppose that f has the first order derivative, which is positive: $f'(x) > 0, \forall x \in [a, b]$. Consider the function $h : [a, b] \to R$

$$h(x) = \frac{f(x)}{\sqrt{f'(x)}}. \tag{1.1}$$

In [2] there is shown that the Halley method for solving

$$f(x) = 0 \tag{1.2}$$

is in fact the Newton method for solving (1.1). This method consists therefore in generating the sequence $(x_n)_{n\geq 0}$ by

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)},\ x_0 \in [a, b],\ n = 0, 1, ...,. \tag{1.3}$$

The first and second order derivatives of h are given by

$$(1.4)\qquad h'(x)=\frac{2[f'(x)]^2-f''(x)f(x)}{2[f'(x)]^{3/2}},\ x\in[a,b]$$

and

$$(1.5)\qquad h''(x)=\frac{[3[f''(x)]^2-2f'''(x)f'(x)]f(x)}{4[f'(x)]^{5/2}},\ x\in[a,b].$$

These relations imply

$$(1.6)\qquad h'(\overline{x})=[f'(\overline{x})]^{1/2}$$

and

$$(1.7)\qquad h''(\overline{x})=0$$

where $\overline{x}\in[a,b]$ denotes the solution of (1.2). As shown in [1], equality (1.7) characterizes the Halley method, in the sense that ensures its convergence order 3. The authors of [4], analyzing an algorithm of Heron for approximating $\sqrt[3]{100}$, give a general algorithm which can be used for approximating the cubic root of any real positive number.

In [7] it is shown that the algorithm from [4] is nothing else than the chord method applied to equation $h(x)=0$ where $h(x)=\frac{f(x)}{\sqrt{f'(x)}}$, with $f(x)=x^3-N$. In this case the equation $h(x)=0$ has the form $x^2-\frac{N}{x}=0$, when $N>0$, $N\in R$.

It is clear that between the Heron algorithm and the Halley method there exists a connection, in the sense that the transformed equation to which we apply the Newton or the chord method is the same. In [7] and [10], the authors study the convergence and error bounds for the Steffensen and Aitken-Steffensen methods applied to (1.1). In this note we shall study a variant of the Aitken-Steffensen method, which differs from those presented in [10] and [11].

We shall consider other two equations, equivalent to (1.2), having the form

$$(1.8)\qquad x-\varphi_1(x)=0$$

and

$$(1.9)\qquad x-\varphi_2(x)=0,$$

where $\varphi_1, \varphi_2 : [a,b] \to [a,b]$ will be conveniently chosen.

We shall study the sequence $(x_n)_{n\geq 0}$ given by

$$(1.10)\quad x_{n+1} = \varphi_1(x_n) - \frac{h(\varphi_1(x_n))}{[\varphi_1(x_n), \varphi_2(\varphi_1(x_n)); h]}, \ x_0 \in [a,b], \ n = 0, 1, \dots .$$

We shall consider the following assumptions on f, φ_1 and φ_2:

i. $f \in C^4[a,b]$;

ii. equation (1.2) has a solution $\overline{x} \in (a,b)$;

iii. $f'(x) > 0, \forall x \in [a,b]$;

iv. φ_1 obeys $0 < [x,y;\varphi_1] < 1, \forall x,y \in [a,b]$, where $[x,y;\varphi_1]$ denotes the first order divided difference of φ_1 on x and y:

$$[x,y;\varphi_1] = (\varphi_1(y) - \varphi_1(x)) / (y-x);$$

v. φ_2 obeys $-1 < [x,y;\varphi_2] < 0, \ \forall x,y \in [a,b]$.

2. The local convergence and error bounds

We shall use the following identities:

$$(2.1)\quad \varphi_1(x_n) - \frac{h(\varphi_1(x_n))}{[\varphi_1(x_n), \varphi_2(\varphi_1(x_n)); h]} = \varphi_2(\varphi_1(x_n)) - \frac{h(\varphi_2(\varphi_1(x_n)))}{[\varphi_1(x_n), \varphi_2(\varphi_1(x_n)); h]}$$
$$n = 0, 1, \dots, .$$

and also the Newton identity

$$(2.2)\qquad h(x) = h(y) + [y,z;h](x-y) + [x,y,z;h](x-y)(x-z)$$

where $[x,y,z;h]$ is the second order divided difference of h on x,y,z.

We notice that equality (1.6) and hypothesis i. ensure the existence of $\alpha, \beta \in R$, $a \leq \alpha < \overline{x} < \beta \leq b$ such that $h'(x) > 0 \ \forall x \in [\alpha, \beta]$.

The following theorem holds:

Theorem 1 *Let $[\alpha,\beta] \subseteq [a,b]$ be such that $h'(x) > 0 \ \forall x \in [\alpha,\beta]$. If the functions f, φ_1, φ_2 and the initial approximation x_0 satisfy:*

a) $x_0 \in [\alpha,\beta]$ can be chosen such that $\varphi_1(x_0) \in [\alpha,\beta]$ and $\varphi_2(\varphi_1(x_0)) \in [\alpha,\beta]$;

b) the hypotheses i-v an satisfied.

Then the following properties are true:

j. for all $n \in N$ we have

$$|x_{n+1} - \overline{x}| \leq \max\{|x_{n+1} - \varphi_1(x_n)|, |x_{n+1} - \varphi_2(\varphi_1(x_n))|\};$$

jj. there exists $k > 0$, $k \in R$, which does not depend on n, such that

$$|x_{n+1} - \overline{x}| \leq k|x_n - \overline{x}|^3, \ \forall n \in N;$$

jjj. if x_0 is close enough to $\overline{x}$ to obey $\sqrt{k}|\overline{x} - x_0| < 1$, then the sequences $(x_n)_{n\geq 0}$, $(\varphi_1(x_n))_{n\geq 0}$ and $(\varphi_2(\varphi_1(x_n)))_{n\geq 0}$ converge to their common limit $\overline{x}$.

P roof. all analyses two cases.

I. $x_0 < \overline{x}$. Then $\varphi_1(x_0) - \overline{x} = \varphi_1(x_0) - \varphi_1(\overline{x}) = [x_0, \overline{x}; \varphi_1](x_0 - \overline{x}) < 0$, i.e., $\varphi_1(x_0) < \overline{x}$. Denote $\Psi(x) = x - \varphi_1(x)$, and so $\Psi(x_0) - \Psi(\overline{x}) = [x_0, \overline{x}; \Psi](x_0 - \overline{x}) = [1 - [x_0, \overline{x}, \varphi_1]](x_0 - \overline{x}) < 0$, i.e. $x_0 < \varphi_1(x_0)$. Now we show that $\varphi_2(\varphi_1(x_0)) > \overline{x}$. From $\varphi_1(x_0) < \overline{x}$ it follows $\varphi_2(\varphi_1(x_0)) - \overline{x} = \varphi_2(\varphi_1(x_0)) - \varphi_2(\overline{x}) = [\overline{x}, \varphi_1(x_0); \varphi_2](\varphi_1(x_0) - \overline{x}) > 0$, i.e. $\varphi_2(\varphi_1(x_0)) > \overline{x}$. Next, we show that $x_1 \in [\varphi_1(x_0), \varphi_2(\varphi_1(x_0))]$, where x_1 is obtained from (1.10) for $n = 0$. Since $h'(x) > 0, \forall x \in [\alpha, \beta]$, and $\varphi_1(x_0) \in [\alpha, \beta]$, we get that $h(\varphi_1(x_0)) < 0$ (we know that $\varphi_1(x_0) < \overline{x}$) and so $x_1$satisfies $x_1 > \varphi_1(x_0)$. We have used the fact that $h'(x) > 0 \ \forall x \in [\alpha, \beta]$ implies $[\varphi_1(x_0), \varphi_2(\varphi_1(x_0)); h] > 0$. Now we show that $x_1 < \varphi_2(\varphi_1(x_0))$. This inequality follows from $\varphi_2(\varphi_1(x_0)) > \overline{x}$, $h(\varphi_2(\varphi_1(x_0))) > 0$ and from (2.1) for $n = 0$. we have shown that

$$x_0 < \varphi_1(x_0) < \overline{x} < \varphi_2(\varphi_1(x_0)) \tag{2.3}$$

and

$$x_1 \in (\varphi_1(x_0), \varphi_2(\varphi_1(x_0))). \tag{2.4}$$

II. $x_0 > \overline{x}$. Similarly to the above reason, we get that

$$x_0 > \varphi_1(x_0) > \overline{x} > \varphi_2(\varphi_1(x_0)) \tag{2.5}$$

and

$$x_1 \in (\varphi_2(\varphi_1(x_0)), \varphi_1(x_0)). \tag{2.6}$$

Denoting by I_0 the open interval determined by $\varphi_1(x_0)$ and $\varphi_2(\varphi_1(x_0))$, then obviously relations (2.3) - (2.6) may be sinthetized as

$$x_1, \overline{x} \in I_0.$$

It can be easily seen that if we denote by I_1 the open interval determined by $\varphi_1(x_1)$ and $\varphi_2(\varphi_1(x_1))$ then get

$$I_1 \subset I_0$$

and

$$x_2, \overline{x} \in I_1$$

where x_2 is obtained from (1.10) for $n = 1$.

Let I_n be the open interval determined by $\varphi_1(x_n)$ and $\varphi_2(\varphi_1(x_n))$ for some $n \in N$. Then repeating the above reason, we may show that

$$\overline{x}, x_{n+1} \in I_n \tag{2.7}$$

and

$$I_{n+1} \subset I_n,$$

where I_{n+1} is determined by $\varphi_1(x_{n+1})$ and $\varphi_2(\varphi_1(x_{n+1}))$. From the above reason and from (2.7) it follows j., which yields an error bound bound for each iteration step.

For jj. using identity (2.2) we get

$$h(\overline{x}) = h(\varphi_1(x_n)) + [\varphi_1(x_n), \varphi_2(\varphi_1(x_n)); h](\overline{x} - \varphi_1(x_n)) + [\varphi_1(x_n), \varphi_2(\varphi_1(x_n)), \overline{x}; h](\overline{x} - \varphi_1(x_n))(\overline{x} - \varphi_2(\varphi_1(x_n)))$$

whence, taking into account (1.10) and $h(\overline{x}) = 0$, we get

$$\overline{x} - x_{n+1} = \frac{[\varphi_1(x_n), \varphi_2(\varphi_1(x_n)), \overline{x}; h]}{[\varphi_1(x_n), \varphi_2(\varphi_1(x_n)); h]}(\overline{x} - \varphi_1(x_n))(\overline{x} - \varphi_2(\varphi_1(x_n))). \tag{2.8}$$

The mean value formulae for the divided differences lead us to

$$[\varphi_1(x_n), \varphi_2(x_n); h] = h'(\theta_n), \theta_n \in I_n \tag{2.9}$$

and

$$[\varphi_1(x_n); \varphi_2(\varphi_1(x_n)), \overline{x}; h] = \frac{h''(\eta_n)}{2}, \eta_n \in I_n.$$

From i. and using the Lagrange formula it follows

$$h''(\eta_n) = h''(\eta_n) - h''(\overline{x}) = h'''(\xi_n)(\eta_n - \overline{x}), \eta_n \in I_n. \tag{2.10}$$

Denoting

$$m_1 = \inf_{x \in [\alpha,\beta]} |h'(x)|$$

and

$$M_1 = \sup_{x \in [\alpha,\beta]} |h'''(x)|,$$

from (2.8) and taking into account (2.9) and (2.10) we get

$$|\overline{x} - x_{n+1}| \leq \frac{M_1}{2m_1} |\overline{x} - \varphi_1(x_n)| \, |\overline{x} - \varphi_2(\varphi_1(x_n))| \, |\overline{x} - \eta_n|.$$

The property jj. follows easily by denoting $k = \frac{M_1}{2m_1}$ and taking into account iv, v, and the fact that $\eta_n \in I_n$.

Property jjj is an immediate consequence of j and jj. ■

3. Determining the functions φ_1 and φ_2

We shall present a modality of choosing φ_1 and φ_2 in order to obey the assumptions of Theorem 2.1

Suppose that f is strictly convex on $[a,b]$, i.e. $f''(x) > 0, \forall x \in [a,b]$. This assumption, together with $f'(x) > 0$ $\forall x \in [a,b]$, lead, by (1.4) to $h'(x) > 0$ $\forall x \in [a,\overline{x}]$. Relation (1.4) again and $f'(x) > 0$ and $f(\overline{x}) = 0$ imply the existence of β, $\overline{x} < \beta \leq b$ such that $h'(x) > 0$, $\forall x \in [\overline{x},\beta]$. These hypotheses ensure the existence of an interval $[\alpha,\beta]$ for which $h'(x) > 0$, $\forall x \in [\alpha,\beta]$. Since $f''(x) > 0$ it follows that $f'(x)$ is increasing on $[a,b]$.

Taking

$$\varphi_1(x) = x - \frac{1}{\mu} f(x)$$

and

$$\varphi_2(x) = x - \frac{1}{\lambda} f(x),$$

with $\mu \geq f'_s(b)$ and $0 < \lambda \leq f'_d(a)$, and assuming that $0 < f'(x) < 2\lambda, \forall x \in [a,b]$, then the functions φ_1 and φ_2 defined above obey hypotheses iv and v of Theorem 2.1. For $a \leq x_0 \leq \overline{x}$ in Theorem 2.1, hypothesis $\varphi_1(x_0) \in [\alpha,\beta]$ is automatically verified, but the assumption $\varphi_2(\varphi_1(x_0)) \in [\alpha,\beta]$ must be kept.

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Received: 10. 03. 2001

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