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SOME CONTRIBUTIONS TO THE THEORY OF TOPOGENOUS SPACES

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Abstract. This paper determines a topogenous structure generated by a topogenous space and a family of sets.

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1. Introduction

The concept of a topogenous space has been introduced in the monograph ([1]) in order to give common generalisation of the concepts of topological space and proximity spaces.

2. Preliminaries and basic definitions

2.1. Definition. A topogenous order on a set X is a binary relation defined for the subset of X denoted by < and which satisfy the following conditions:

a) $\theta < \theta$ and $X < X$;

b) $A < B$ implies $A \subset B$;

c) $A < B \subset D$ implies $A < D$;

d) $A_1 < B_1$, $A_2 < B_2$ implies $A_1 \cap A_2 < B_1 \cap B_2$;

e) $A_1 < B_1$, $A_2 < B_2$ implies $A_1 \cup A_2 < B_1 \cup B_2$.

2.2. Definition. A topogenous structure on X is a topogenous order on X that is idempotent.

2.3. Exemples.

1) If τ is a topology on set and we define $A <_\tau B$ if and only if $A \subset \text{int}B$ then $<_\tau$ is a topogenous structure on X.

2) If δ is a proximity relation on X and we can define $<_\delta$ like:

$A <_\delta B$ if and only $A \,\delta\, (X - B)$ then $<_\delta$ is a topogenous and structure on X

2.4. Definition. A topogenous space is a pair (X, <) where X is a set and < is a topogenous structure on X.

2.5. Definition. If $(X,<_1)$ and $(Y,<_2)$ are topogenous spaces and $f: X \to Y$ is an application, then f is named continuous if and only if $A <_2 B$ implies $f^{-1}(A) <_1 f^{-1}(B)$.

2.6. Notations. Let us denote by TPG the category in which the topogenous spaces as objects, and $(<_1, <_2)$ – continuous maps as morphisms and by TOP the category in which the topological spaces as objects and continuous maps and morphism.

2.7. Definitions. The topogenous order < is perfect if and only if:

$$A_i < B_i (i \in J) \text{ implies } U\{A_i | i \in J\} < U\{B_i | i \in J\}$$

2.8. Remark. The category TOP can be isomorphically embedded by means of 2.3. into TPG. A topogenous space (X, <) is an object of TOP if the topogenous order < is perfect.

2.9. Definition. Let (X, <) be a topogenous spaces and A a family of subset of X. The family A equipped with the order < is named topogenous – order. The family (A, <) is named m – inductively order if any subfamily of A have a minimun.

2.10. Notation. If (X, <) is a topogenous space then we denote with $D\tau_<$ the family:

$$D\tau_< = \{G | G < G\}, \ G \subset X.$$

3. Topogenous spaces and topological spaces

3.1. Proposition. Let (X, <) be a topogenous space and S a family of subset of X closed with respect to the intersection. Let $S \in S$, $O \in D_{\tau<}$ and F (O, S) is the finest filter on X with respect to the property:

$$O \in F(O, S) \text{ implies } O < S.$$

If $S \cap D\tau_< = \theta$ and F (O,S) is m – inductively ordered the there is a family F (G, H), where $G \in D\tau_<$ and $H \in S$, that such:

1) (F (G,H), <) is m – inductively order;
2) G is the minimal elemnt of the family (F (G,H), <)

Proof. Let $O \in D_{\tau<}$, $S \in S$ and F (O, S) the finest filter on X with respect to properties: F (O, S) is m – inductively order and $O \in F(O, S)$ implies $O < S$. Considering the family:

$$J = \{(F(O_a, S_a), <) | a \in A\}$$

ordered through inclusion, where $O_a \in D\tau_<$, $S_a \in S$ and J' a subfamily of J totally ordained so:

$$J' = \{(F(O_b, S_b), <), b \in B \subset A\}$$

Let:

$$G' = U\{O_b | b \in B\}, \quad \cap\{S_b | b \in B\} = H'$$

From

$$G' \supset O_b \in F(O_b, S_b) \ (\forall) b \in B$$

$$\underset{b \in B}{U} O_b < \underset{b \in B}{\cap} S_b \subset S_b$$

we get:

$$\underset{b \in B}{U} O_b < S_b$$

and hence:

$$G' \in \mathrm{F}(O_b, S_b) \ (\forall) b \in B \subset A$$

From the definition of the family $\mathrm{F}(G', H')$ we deduce that the inclusion:

$$\mathrm{F}(O_b, S_b) \subseteq \mathrm{F}(G', H') \ (\forall) b \in B$$

will take place and using Zorn's lemma we deduce that the family J has a maximum element ($\mathrm{F}(G, H)$, <) in relation with the inclusion and which is m – inductively ordered, i.e. 1) was demonstrated.

Let $T \in (\mathrm{F}(G, H), <)$ and considering the family {G. T}. Taking into consideration that ($\mathrm{F}(G, H)$, <) is m – inductively ordered we deduce that {G, T} has a minimum. If T< G then having in view the relation G < H and 2.1. (b) and c) we deduce that T < H. We may write:

$$T < H \ (\forall) T \in (\mathrm{F}(G, H), <)$$

If $T = H \in (\mathrm{F}(G, H), <)$ we obtain the relation H < H will take place, hence H∈Dτ<. On the other hand H∈S, hence H∈S∩Dτ<, therefore we obtain S∩ Dτ< ≠ 0, which is not true. Hence G is the minimal element of the family $(\mathrm{F}(G, H), <)$

3.2. Proposition. Let (X, <) be a topogenous space and S a family of nonemty subset of X such that for any S∈S and O∈Dτ< the family (F (O, S), <) is closed in relation with the finite intersection. Then the following equality takes place:

$$\mathrm{F}(O_1 \cap O_2, S_1) \cap \mathrm{F}(O_1 \cap O_2, S_2) = \mathrm{F}(O_1 \cap O_2, S_1 \cap S_2)$$

Prof. From $O_1 \cap O_2 < S_1$, $O_1 \cap O_2 < S_2$ and 2.1. (d) we deduce that $O_1 \cap O_2 < S_1 \cap S_2$. We obtain the inclusion:

$$\mathrm{F}(O_1 \cap O_2, S_1) \cap \mathrm{F}(O_1 \cap O_2, S_2) \subseteq \mathrm{F}(O_1 \cap O_2, S_1 \cap S_2)$$

From $O_1 \cap O_2 < S_1 \cap S_2 \subset S_1$ we get that $O_1 \cap O_2 < S_1$ and we may write: $\mathrm{F}(O_1 \cap O_2, S_1 \cap S_2) \subseteq \mathrm{F}(O_1 \cap O_2, S_1)$

but this shows that:

$$O_1 \cap O_2 < S_1 \cap S_2 \subset S_2 \text{ implies } O_1 \cap O_2 < S_2$$

In conclusion:

$$\mathrm{F}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) \subseteq \mathrm{F}(\mathrm{O}_1 \cap O_2, S_2)$$

and hence we deduce:

$$\mathrm{F}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) \subseteq \mathrm{F}(\mathrm{O}_1 \cap O_2, S_1) \cap \mathrm{F}(\mathrm{O}_1 \cap O_2, S_2)$$

and the propozition is proved.

3.3. Notation. Let (E, <) be a topogenous space and S a family of nonempty subset of set E which is closed in relation with the reunion and finite intersection. For each S∈S and O∈Dτ< we definie the set f(O,S) as being the maximum element of the family m-inductively orderer (F (O, S), <).

3.4. Propozition. Let (X, <) be a topogenous space, S a family of subset of set X, which is closed in relation to the finite intersection. Then the following equality will take place:

$$\mathrm{f}(\mathrm{O}_1 \cap O_2, S_1) \cap f(\mathrm{O}_1 \cap O_2, S_2) = f(\mathrm{O}_1 \cap O_2, S_1 \cap S_2)$$

$$(\forall) S \in \mathrm{S},\ (\forall) O_i \in D\tau_{<},\ i = 1, 2$$

Proof. From:

$$\mathrm{f}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) \in \mathrm{F}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2)$$

and by 3.2. we get:

$$\mathrm{f}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) \in \mathrm{F}(\mathrm{O}_1 \cap O_2, S_1) \cap \mathrm{F}(\mathrm{O}_1 \cap O_2, S_2)$$

On the other hand we have:

$$\mathrm{f}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) \in \mathrm{F}(\mathrm{O}_1 \cap O_2, S_i),\quad i = 1, 2$$

Therefore we have:

$$\mathrm{f}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) < f(\mathrm{O}_1 \cap O_2, S_i),\quad i = 1, 2$$

hence we hand:

$$\mathrm{f}(\mathrm{O}_1 \cap O_2, S_1 \cap S_2) \subset f(\mathrm{O}_1 \cap O_2, S_i),\quad i = 1, 2.$$

This shows that:

$$f(O_1 \cap O_2, S_1 \cap S_2) \subset f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)$$

On the other hand we have:

$$f(O_1 \cap O_2, S_1 \cap S_2) \in F(O_1 \cap O_2, S_1 \cap S_2)$$

$$f(O_1 \cap O_2, S_1) \in F(O_1 \cap O_2, S_1 \cap S_2)$$

$$f(O_1 \cap O_2, S_2) \in F(O_1 \cap O_2, S_1 \cap S_2)$$

$$f(O_1 \cap O_2, S_1 \cap S_2) < f(O_1 \cap O_2, S_1)$$

$$f(O_1 \cap O_2, S_1 \cap S_2) < f(O_1 \cap O_2, S_2)$$

Consequently, we have:

$$f(O_1 \cap O_2, S_1 \cap S_2) < f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)$$

$$f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2) \in F(O_1 \cap O_2, S_1 \cap S_2)$$

and by hypothesis, < is idempotent and hence there exist a set

$$D \in F(O_1 \cap O_2,\ S_1 \cap S_2)$$

such that:

$$f(O_1 \cap O_2, S_1 \cap S_2) < D < f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)$$

Now, put:

$$R = \{L \in F(O_1 \cap O_2, S_1 \cap S_2) \mid f(O_1 \cap O_2, S_1 \cap S_2) < L < f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)\}$$

and from:

$$R \subset F(O_1 \cap O_2, S_1 \cap S_2)$$

because $F(O_1 \cap O_2, S_1 \cap S_2)$ is m – inductively ordered we deduce that R has a minimum

$D_0 \in F(O_1 \cap O_2, S_1 \cap S_2)$, i.e.

$$f(O_1 \cap O_2, S_1 \cap S_2) < D_0 < f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)$$

The order < is idempotent, hence there exists the set D_0' such that:

$$D_0' \in F(O_1 \cap O_2, S_1 \cap S_2)$$

$$f(O_1 \cap O_2, S_1 \cap S_2) < D_0' < D_0 < f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)$$

which is a contradiction. This shows that:

$$f(O_1 \cap O_2, S_1 \cap S_2) = f(O_1 \cap O_2, S_1) \cap f(O_1 \cap O_2, S_2)$$

and thus the proposition was demonstrated.

3.5. Proposition. Let (X, <) be a topogenous space, S a family of subset of set X, which is closed in relation to the finite intersection. Then the family:

$$\Omega_\varphi^<(Q) = \{O \cap f(Q, S) | O \in D\tau_<, S \in \mathrm{S}\}$$

is the sub – base of a topology wich is defined on E, where Q $\in D\tau_<$ is a given set.

Proof. Let $Y_1, Y_2 \in \Omega_\varphi^<(Q)$ be two arbitrary sets. In this case we have:

$$Y_1 = O_1 \cap f(Q, S_1), \quad Y_2 = O_2 \cap f(Q, S_2)$$
$$O_1, O_2, Q \in D\tau_<, \qquad S_1, S_2 \in \mathrm{S}$$
$$Y_1 \cap Y_2 = O_1 \cap O_2 \cap f(Q, S_1) \cap f(Q, S_2)$$

Consequetly, from proposition 3.4., we obtain:

$$f(Q, S_1) \cap f(Q, S_2) = f(Q \cap Q, S_1) \cap f(Q \cap Q, S_2) =$$
$$= f(Q \cap Q, S_1 \cap S_2) = f(Q, S_1 \cap S_2)$$

This shows that $\Omega_\mathrm{S}^<(Q)$ is the sub – base of a topology which is defined on X.

3.6. Definition. The topology with the sub – base $\Omega_\mathrm{S}^<(Q)$ is denoted by $\tau_<^\mathrm{S}(Q)$ and it is named the topology generated by S and topogenous structure < relative to set Q.

3.7. Notation. The family of all open sets in relation with the topology $\tau_<^\mathrm{S}(Q)$ is denoted by $D\tau_<^\mathrm{S}(Q)$.

3.8. Proposition. Let (E, <) be a object of TPG, S a family of subset of E closed with respect to intersection, G $\in D\tau_<^\mathrm{S}(Q)$ and Q $\in D\tau_<$. Then for every n $\in$ N there exists the set $G_j \in D\tau_<^\mathrm{S}(Q)$, $j = \overline{1, n}$ and $R_j \in$ S such that:

$$G=\bigcup_{j=1}^{n}\{G_j\cap f(Q,R_j)\}$$

Proof. Let G $\in D\tau_c^S(Q)$. In this cas we have:

$$G=\bigcup_{j=1}^{n}\left[\bigcap_{i=1}^{j}Q_i\cap f(Q,S_i)\right]$$

where Q $\in D\tau_c$ and $S_i \in S(i=\overline{1,n})$. By using proposition 3.5. , we obtain:

$$G=\bigcup_{j=1}^{n}\left[\left(\bigcap_{i=1}^{j}Q_i\right)\cap f\left(Q,\bigcap_{i=1}^{j}S_i\right)\right]$$

Now, put:

$$O_j=\bigcap_{i=1}^{j}Q_i,\quad R_j=\bigcap_{i=1}^{j}S_i$$

It is easy to observe that:

$$Q_j\in D\tau_c,\quad R_j\in S,\quad j=\overline{1,n}$$

Thus, it results that we have:

$$G=\bigcup_{j=1}^{n}O_j\cap f(Q,R_j)$$

and this completes the proof.

References

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