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SOME NONCOMMUTATIVE DIFFERENTIAL FORMS

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Abstract. In this paper I will present some aspects about the forms in the noncomutative geometry theory. In the first line, I will present the form called De Rham Commutative form, after that I will present Noncommutative De Rham Form and finally I give a concrete example for this theory.

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1. Introduction. I will start this presentation by remembering some aspects about graded differential algebras.

Consider a manifold M, the K-algebra unital, 1 being the element one, of smooth differential functions $\mathbf{C}_K^\infty(M)$ and the $\mathbf{C}_K^\infty(M)$ –module $\chi(M)$ of vector fields, a 1-differential form over M is an element of the dual module $Hom_{\mathbf{C}_K^\infty(M)}(\chi(M), \mathbf{C}_K^\infty(M)) = \chi(M)^*$ which is usually written $\Omega_K^1(M)$,it is a symmetric $\mathbf{C}_K^\infty(M)$–bimodule through : $(f\alpha)(X) = f\alpha(X) = \alpha(X)f$

Definition. A differential is then a K-linear map :

$$d : \mathbf{C}_K^\infty(M) \to \Omega_K^1(M)$$
$$f \mapsto df$$

defined by: $(df)(X) = Xf$ and as it has the property that for any $a, b \in \mathbf{C}_K^\infty(M)$, $d(ab) = (da)b + a(db)$ it is a K-derivation of the K-algebra $\mathbf{C}_K^\infty(M)$ with values in the $\mathbf{C}_K^\infty(M)$-bimodule $\Omega_K^1(M)$. Usually we put $\mathbf{C}_K^\infty(M) = \Omega_K^0(M)$ so that the differential d is a K-linear map $d = d^0 : \Omega_K^0(M) \to \Omega_K^1(M)$

Definition The n-the exterior power of $\Omega_K^1(M)$ is, for any $n > 0$:

$$\Omega_K^n(M) = \bigwedge_{\mathbf{C}_K^\infty(M)}^n \Omega_K^1(M)$$

Definition The exterior product of a two forms: $\alpha = \alpha_1 \wedge ... \wedge \alpha_p \in \Omega_K^p(M)$, $\beta = \beta_1 \wedge ... \wedge \beta_q \in \Omega_K^q(M)$ is: $\alpha \wedge \beta = \alpha_1 \wedge ... \wedge \alpha_p \wedge \beta_1 \wedge ... \wedge \beta_q$

Definition A graded K-algebra is $\Omega_K^*(M) = \bigoplus_{n \in N} \Omega_K^n(M)$

And the last definition for this introduction part:

Definition A graded differential over the graded R-algebra $A^* = \bigoplus_{n \in Z} A$ with Z as the grading set, is a R-linear map $d^n : A^n \to A^{n+1}$ with properties:

1. $(d^{n+1} \circ d^n)(\alpha) = 0 \quad for\ \alpha \in \Omega_a^n(A)$
2. $d(ab) = (da)b + (-1)^{gr(a)} a(db) \quad for\ a, b \in A^*$

2. Commutative De Rham Forms

Let R be an unital commutative ring and let be given an associative and ambidextrous R-algebra A, the algebra $A \otimes_R A$ is a A-bimodule for the left and right actions given by:

$$a(x \otimes y) = (ax) \otimes y,\ (x \otimes y)a = x \otimes (ya)$$

The multiplication map $\mu : A \otimes_R A \to A$, that is given by $\mu(a \otimes b) = ab$ is bimodule morphism and we have homogeneity:

$$\mu(a(x \otimes y)b) = \mu((ax) \otimes (yb)) = (ax)(yb) = a(xy)b = a\mu(x \otimes y)b$$

the kernel of μ is I , and assuming that R contains Q,we obtain , if $a \otimes b \in I$, i.e. ab=0, then:

$$a \otimes b = \frac{1}{2}(a(1 \otimes b - b \otimes 1) - (1 \otimes a - a \otimes 1)b)$$

so that I is generated by elements of the form $1 \otimes a - a \otimes 1$.

In general the multiplication map μ is not an R-algebra morphism, because:

$$\mu((a \otimes b)(a' \otimes b')) = \mu(aa' \otimes bb') = (aa')(bb')$$
$$\mu(a \otimes b)\mu(a' \otimes b') = (ab)(a'b')$$

In the case if the R-algebra A is commutative so that then I= ker μ, is an R ideal of $A \otimes_R A$ and taking the ideal I^2 the quotient is a symmetric bimodule, suppose now that the R-algebra A is commutative and unital and contains Q, with element one 1. The application :

$$d : A \to I/I^2$$
$$a \to 1 \otimes a - a \otimes 1$$

is the universal derivation. We put $\Omega_a^1 = I/_{I^2}$, where the letter a is for the abelian and call its elements the commutative de Rham 1-differential forms over the commutative unital R-algebra of A.

In general the commutative De Rham n-differential forms over the commutative unital R-algebra A are defined in this way:

$$\Omega_a^n(A) = \bigwedge^n \Omega_a^1(A)$$

and it can be shown that this bimodule is generated by elements of the form $a_0(da_1)\wedge \ldots \wedge (da_n)$,and we can define a product:

$$(a_0(da_1)\wedge \ldots \wedge (da_p)) \wedge (b_0(db_1)\wedge \ldots \wedge (db_q))$$

This way we get a graded R-algebra: $\Omega_a^{\bullet}(A) = \otimes_{n\in Z} \Omega_a^n(A)$

Now we will define a map in this way:

$$d : \Omega_a^{\bullet}(A) \rightarrow \Omega_a^{n+1}(A)$$

$$d(a_0(da_1)\wedge \ldots \wedge (da_p)) = (da_0)\wedge (da_1)\wedge \ldots \wedge (da_p)$$

and with this definition we obtain a graded differential R-algebra $\Omega_a^{\bullet}(A)$, called the R-algebra of commutative de Rham differential forms over the commutative unital R-algebra A. We have in fact:

1. $(d^{n+1} \circ d^n)(\alpha) = 0 \quad for\ \alpha \in \Omega_a^n(A)$
2. $d(\alpha \wedge \beta) = (d\alpha)\wedge \beta + (-1)^{gr(\alpha)} \alpha \wedge (d\beta) \quad for\ \alpha, \beta \in \Omega_a^{\bullet}(A)$

3. Noncommutative De Rham Differential Forms

Let R be an unital commutaive ring containing Q and let be given an associative and ambidextrous R-algebra A, the kernel I of $\mu : A \otimes_R A \rightarrow A$ is a A-bimodule , and the element $1 \otimes x - x \otimes 1$ generate I. We use the notation $\Omega_{na}^1(A)$ for that bimodule I, where the letters na stand for non abelian , and we can define a derivation of A with values in $\Omega_{na}^1(A)$ through:

$$d : A \rightarrow \Omega_{na}^1(A)$$

$$a \rightarrow 1 \otimes a - a \otimes 1$$

getting this way a universal derivation.

$\Omega_{na}^1(A)$ is called the A-bimodule of noncomutative De Rham differential 1-forms over the unital algebra of A, and now we can write the left or the right action of a on $\Omega_{na}^1(A)$,in the next way: $a(db) = adb$,and for the right action we have:

$$(db)a = d(ba) - b(da)$$

where for $a(db)$ we write $a \otimes b - ab \otimes 1$.

We can rephrase in this way: $(1 \otimes b - b \otimes 1)a = 1 \otimes ba - b \otimes a$.

The fact that $\Omega^1_{nc}(A)$ is a A-bimodule allows to write for $\omega \in \Omega^1_{nc}(A)$:

$$a(\omega b) = (a\omega)b.$$

More generally the noncommutative de Rham differential n-forms over the unital R-algebra A are defined through the n-th tensor power of the A-bimodule $\Omega^1_{nc}(A)$:

$$\Omega^n_{nc}(A) = \bigotimes_A^n \Omega^1_{nc}(A) = \Omega^1_{nc}(A) \otimes_A ... \otimes_A \Omega^1_{nc}(A)$$

Let analyze the case n = 2.

In this case the generators of $\Omega^2_{nc}(A) = \Omega^1_{nc}(A) \otimes \Omega^1_{nc}(A)$ have the next form: $a_0(da_1) \otimes b_0(db_1)$, so we can write:

$$a_0(da_1) \otimes b_0(db_1) = a_0(da_1)b_0 \otimes db_1 = a_0(d(a_1b_0) - a_1(db_0)) \otimes db_1 =$$

$$= a_0(da_1b_0) \otimes db_1 - a_0a_1(db_0) \otimes db_1$$

and we remark that $\Omega^2_{nc}(A)$ is generated by the elements with the form: $c_0dc_1 \otimes dc_2$, in general $\Omega^n_{nc}(A)$ is generated by the elements with the form: $c_0dc_1 \otimes dc_2 \otimes ... \otimes dc_n$.

Now we define a product in $\Omega^\bullet_{nc}(A) = \bigoplus_{n=0}^{\infty} \Omega^n_{nc}(A)$, where $\Omega^0_{nc}(A) = A$ with the folowing motivation: to multiply a 1-form by a 1-form we could proceed the product in this way:

$$(a_0da_1)(a_2da_3) = (a_0da_1)a_2 \otimes da_3 =$$

$$= ((a_0da_1)a_2)) \otimes da_3 = a_0d(a_1a_2) \otimes da_3 - a_0a_1da_2 \otimes da_3$$

Now for the product of a n-form by a k-form we define their product as a (n+k) form:

$$(a_0(da_1) \otimes ... \otimes (da_n))(a_{n+1}(da_{n+2}) \otimes ... \otimes (da_{n+k+1}) =$$

$$= \sum_{i=1}^{n} (-1)^{n-i} a_0(da_1) \otimes ... \otimes da_n \otimes da_{n+1} \otimes da_{n+2} \otimes ... \otimes da_{n+k-1} +$$

$$+ (-1)^n a_0a_1da_2 \otimes ... \otimes da_{n+k+1}$$

For the particular case n=1 and k=1, and $a_0 = a_2 = 1$ we get: $da_1da_3 = da_1 \otimes da_3$.

More general, we will have: $a_0(da_1)...(da_n) = a_0(da_1) \otimes ... \otimes (da_n)$.

Moreover we can define: $d(a_0(da_1)...(da_p)) = (da_0)(da_1)...(da_p)$
And with this definition we have:

$$(d^{n+1} \circ d^n)(\alpha) = 0 \quad for \quad \alpha \in \Omega^n_{na}(A)$$
$$d(\alpha\beta) = (d\alpha)\beta + (-1)^{gr(\alpha)}\alpha(d\beta) \quad for\ \alpha, \beta \in \Omega^n_{na}(A),$$

so we get a graded differential R-algebra called the R-algebra $\Omega^{\bullet}_{na}(A)$ of noncommutative differential forms over the unital algebra A with the differential

$$d : \Omega^n_{na}(A) \to \Omega^{n-1}_{na}(A).$$

4. Example

In the next lines I developed one particular example for this theory:
We consider the quaternion algebra:

$$H = R + iR + jR + kR$$

where $i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik$ the element $x = \alpha_0 + i\alpha_1 + j\alpha_2 + k\alpha_3$ it is called quaternion.
We consider now one model for the quaternion algebra:
$\Phi(\mathbf{H}) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; \quad a, b \in \mathbf{C} \right\} \subseteq M_2(\mathbf{C})$ where $\Phi : \mathbf{H} \to \Phi(\mathbf{H})$ with this consideration we can associate to each quaternion $x \in \mathbf{H}$, the following:

$$\Phi_x = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \alpha_0 q_0 + \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3$$

with $q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

We denote with f, the Φ_x matrix.
And now we can compute for this algebra the differential df, in this way:
$df = 1 \otimes f - f \otimes 1$ and we will obtain:

$$1 \otimes f = \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix}$$

and after this we compute

$$f\otimes 1=\begin{pmatrix} a & 0 & b & 0\\ 0 & a & 0 & b\\ -\bar{b} & 0 & \bar{a} & 0\\ 0 & -\bar{b} & 0 & \bar{a}\end{pmatrix}$$

and now we obtain the main result: $df=1\otimes f-f\otimes 1=\begin{pmatrix} 0 & b & -b & 0\\ -\bar{b} & \bar{a}-a & 0 & -b\\ \bar{b} & 0 & a-\bar{a} & b\\ 0 & \bar{b} & -\bar{b} & 0\end{pmatrix}$

Taking 2 matrices $f,g\in\Phi(\mathbf{H})$: $f=\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}$ and $g=\begin{pmatrix} c & d\\ -\bar{d} & \bar{c}\end{pmatrix}$ we will get:

$$fdg=\begin{pmatrix} b\bar{d} & bd & b(c-\bar{c})-ad & bd\\ -a\bar{d} & a(\bar{c}-c)+b\bar{d} & -b\bar{d} & -ad\\ \overline{ad} & -\bar{b}d & \bar{a}(c-\bar{c})+\bar{b}d & \bar{a}d\\ \overline{bd} & -\bar{b}(\bar{c}-c)+\overline{ad} & -\overline{ad} & \bar{b}d\end{pmatrix}$$

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