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ON CONGRUENCES ON n-SEMIGROUPS AND ON THEIR BINARY REDUCES

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Abstract. Let $(A,()_\circ)$ be an n-semigroup $n \geq 3$ with right unit u_1^{n-1} and ρ an equivalence relation on the set A .

1. ρ is a congruence relation on the n-semigroup $(A,()_\circ)$ iff for all $a,b \in A$ and for every sequence c_1^{n-1} over A the following statements hold:
$a\rho b \Longrightarrow (c_1, a ,c_2^{n-1})_\circ \rho (c_1,b ,c_2^{n-1})_\circ$ and $(a ,c_1^{n-1})_\circ \rho (b ,c_2^{n-1})_\circ$;

2. moreover, if $u_{n-1}u_1^{n-2}$ is left unit in A, ρ is a congruence relation of the n-semigroup $(A,()_\circ)$ iff for all $a,b \in A$ and for every sequence c_1^{n-1} over A $a\rho b \Longrightarrow (c_1,a, c_2^{n-1})_\circ \rho\ (c_1, b, c_2^{n-1})_\circ$.

If ρ is a congruence relation of the n-semigroup (n-group) $(A,()_\circ)$ then ρ is a congruence on binary reduce $red_{u_1^{n-2}}(A,()_\circ)(red_c A; c \in A)$. Moreover if u_1^{n-1} is a right unit in $(A,()_\circ)$ and for all $a,b \in A$:
$a\rho b \Longrightarrow (u_{n-1}, a, u_1^{n-2})_\circ \rho (u_{n-1}, b, u_1^{n-2})_\circ ((c, a, \bar{c}, \overset{(n-3)}{c})_\circ \rho (c, b, \bar{c}, \overset{(n-3)}{c})_\circ)$,
then the converse statement is also true.

This results generalize and improve the result of Usan [8],[9] for n-groups.

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1. *Notations.* The sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j.
For $j < i$, x_i^j is the empty symbol. The sequence $\underbrace{x, \ldots x}_{k}$ will be denoted by $\overset{(k)}{x}$.

A set A together with an n-ary operation $()_\circ : A^n \to A$ is called *n-groupoid.* An n-groupoid $(A,()_\circ)$ is called *n-semigroup* if for any $i \in \{2,\ldots,n\}$ and all $x_1, \ldots x_{2n-1} \in A$ (as in [6] we shall use the abbreviated notation: $x_1^{2n-1} \in A$) the following laws hold:

$$\left((x_1^n)_\circ, x_{n+1}^{2n-1}\right)_\circ = \left(x_1^{i-1}, \left(x_i^{i+n-1}\right)_\circ, x_{i+n}^{2n-1}\right)_\circ$$

An $(n-1)$-ad u_1^{n-1} of elements of an n-semigroup A is called *a right unit (left unit)*, if for all $x \in A$ we have $\left(x, u_1^{n-1}\right)_\circ = x(\left(u_1^{n-1}, x\right)_\circ = x$ respectively).

An n-semigroup $(A, ()_\circ)$ is called *n-group* [1] if for any $i \in \{1, 2, \ldots, n\}$ and all $a_1^n \in A$ the equation $\left(a_1^{i-1}, x, a_{i+1}^n\right)_\circ = a_i$ has a unique solution in A. In an n-group the unique solution of the equation $\left(\overset{(n-1)}{a}, x\right)_\circ = a$ is called *the querelement of "a"* and it is denoted by $\overline{a}$. In this case the $(n-1)$-ad $\overset{(i-2)}{a}\ \overline{a}\ \overset{(n-i)}{a}$ is a right unit and a left unit, for $\forall a \in A$ and $i \in \{2, \ldots, n\}$.

Let $(A, ()_\circ)$be an n-semigroup and $u_1^{n-2} \in A$ arbitrary fixed elements of A. The structure $(A, \cdot)$ where

$$x \cdot y = \left(x, u_1^{n-2}, y\right)_\circ \tag{1.1}$$

is a semigroup, called *the binary reduce of A with respect* to the elements u_1^{n-2} ([3], [10]); it is denoted $red_{u_1^{n-2}}(A, ()_\circ)$ or simple $red_{u_1^{n-2}} A$.

The binary reduce of an n-group A with respect to the elements $\overset{(n-3)}{a}\ \overline{a}$ is a group denoted by $red_a A$. All the binary reduces of an n-group are isomorphic.

Let $(A, \cdot)$ be a semigroup, $c \in A$ and $\alpha \in End\,(A, \cdot)$. The structure $(A, ()_\circ)$ where the n-ary operation $()_\circ : A^n \to A$ is defined by

$$(x_1^n)_\circ = x_1 \cdot \alpha(x_2) \cdot \ldots \cdot \alpha(x_n) \cdot c, \tag{1.2}$$

is called *the n-ary extension of the semigroup A with respect to the endomorphism α on the element $c \in A$*; it is denoted by $ext_{\alpha,c}(A, \cdot)$.

In [3] is proved that if

$$\alpha^n(x) \cdot \alpha(c) = c \cdot \alpha(x), \forall x \in A, \tag{1.3}$$

then $ext_{\alpha,c}(A, \cdot)$ is an n-semigroup and moreover :

Theorem 1.1. [3] ([2]). *If $u_1^{n-1}(\overset{(n-3)}{a}\ \overline{a})$ is a right unit in the n-semigroup (n-group) $(A, ()_\circ)$, $c = \left(\overset{(n)}{u}_{n-1}\right)_\circ$ (or $c = \left(\overset{(n)}{a}\right)_\circ$) and*

$\alpha : A \to A, \alpha(x) = \left(u_{n-1}, x, u_1^{n-2}\right)_\circ$(or $\alpha(x) = \left(\overline{a} x \overset{(n-3)}{a}\right)_\circ$), then α is an endomorphism (automorphism) of the binary reduce $red_{u_1^{n-2}}(A, ()_\circ)$ (of $red_a(A, ()_\circ)$) and

$$\begin{aligned} &ext_{\alpha,c}\left(red_{u_1^{n-2}}(A, ()_\circ)\right) = (A, ()_\circ); \\ &(ext_{\alpha,c}(red_a(A, ()_\circ)) = (A, ()_\circ)) \end{aligned} \tag{1.4}$$

2. Let $(A, ()_\circ)$ be an n-groupoid and ρ an equivalence relation. If $a_i \rho b_i;\ 1 = \overline{1,n} \Longrightarrow (a_1^n)_\circ \rho (b_1^n)_\circ$ then ρ is called the *congruence relation of n-groupoid*. The following proposition is true: *ρ is a congruence of n-groupoid if ρ is equivalence relation on A and*

$$\forall\, c_1^{n-1} A,\ (a\rho b) \Longrightarrow (c_1^{i-1} a c_i^n)_\circ \rho (c_1^{i-1} b c_i^{n-1})_\circ;\ i = \overline{1,n}.$$

A congruence relation ρ on an n-groupoid $(A, ()_\circ)$ is said to be *normal* iff the following holds

$\forall a \in A,\ \forall b \in A;\ \forall c_1^{n-1} \in A\ (c_1^{i-1}, a, c_i^{n-1})_\circ \rho (c_1^{i-1} b c_i^{n-1})_\circ \Longrightarrow a\rho b;\ i = \overline{1,n}$.

Uşan [8] showed that if $(A, ()_\circ)$ is an n-group, $e : A^{n-2} \to A$ its {1,n}-neutral operation [6], $f : A^{n-1} \to A$ its inverting operation [7] and ρ is a congruence relation of the n-group $(A, ()_\circ)$, then the next statements are equivalent:

1) ρ is a normal congruence on the n-groupoid $(A, ()_\circ)$ for every $n \geq 2$;
2) ρ is a normal congruence of the $(n-2)$–groupoid (A, e) for every $n \geq 3$;
3) ρ is a congruence of the $(n-1)$-groupoid (A, f) for every $n \geq 2$;
4) ρ is a normal congruence of (A, f) for $n = 2$.

In the present note we shall study the congruence relations of n-semigroups with a right unit and the connection between them and those of its binary reduces.

Proposition 2.1. *If $(A, ()_\circ)$ is an n-semigroup with u_1^{n-1} a right unit and ρ an equivalence relation on A satisfying the conditions: for all $a, b \in A$*

$$a\rho b \Longrightarrow (c_1, a, c_2^{n-1})_\circ \rho (c_1, b, c_2^{n-1})_\circ \text{ for all } c_1^{n-1} \in A \tag{2.1}$$

and

$$a\rho b \Longrightarrow (a, c_1^{n-1})_\circ \rho (b, c_2^{n-1})_\circ \text{ for all } c_1^{n-1} \in A, \tag{2.2}$$

then ρ is a congruence relation on $(A, ()_\circ)$;

2. If $(A, ()_\circ)$ is an n-semigroup with u_1^{n-1} a right unit, $u_{n-1} u_1^{n-2}$ a left unit and ρ an equivalence relation on A satisfying the condition (2.1), then ρ is a congruence relation on $(A, ()_\circ)$;

Proof. Let $(A, ()_\circ)$ be an n-semigroup with a right unit u_1^{n-1} and ρ an equivalence relation on the set A satisfying (2.1)

Then, the following statements hold:

$$\forall a, b \in A,\ \forall c_1^{n-1} \in A;\ \left(\bigwedge_{i=3}^{n} \ (a\rho b \Rightarrow (c_1^{i-1} a c_i^{n-1})_\circ \rho (c_1^{i-1} b c_i^{n-1})_\circ \right) \tag{2.3}$$

Indeed, let a, b, c_1^{n-1} be arbitrary elements of the set A such that $a\rho b$. By the assumption (2.1) and associativity of n-ary operation we have the implications:

$$a\rho b \Longrightarrow (c_2 a u_1^{n-2})_\circ \rho (c_2 b u_1^{n-2})_\circ \Longrightarrow$$

$$\Longrightarrow (c_1 (c_2 a u_1^{n-2})_\circ u_{n-1} c_3^{n-1})_\circ \rho (c_1 (c_2 b u_1^{n-2})_\circ u_{n-1} c_3^{n-1})_\circ \Longrightarrow$$

$$\Longrightarrow (c_1 c_2 a c_3^{n-1})_\circ \rho (c_1 c_2 b c_3^{n-1})_\circ.$$

If (2.3) is true for $i = k \in \{3, \ldots, n-1\}$ then (2.3) is true for $i = k+1$ because we have the implications

$$a\rho b \Longrightarrow (c_2^k a u_1^{n-k})_\circ \rho (c_2^k a u_1^{n-k})_\circ \overset{(2.1)}{\Longrightarrow}$$

$$\Longrightarrow (c_1 (c_2^k a u_1^{n-k})_\circ u_{n-k+1}^{n-1} c_{k+1}^{n-1})_\circ \rho (c_1 (c_2^k b u_1^{n-k})_\circ u_{n-k+1}^{n-1} c_{k+1}^{n-1})_\circ \Longrightarrow$$

$$\Longrightarrow (c_1^k a c_{k+1}^{n-1})_\circ \rho (c_1^k b c_{k+1}^{n-1})_\circ.$$

For $k = n-1$ from here we have $a\rho b \Longrightarrow (c_1^{n-1} a)_\circ \rho (c_1^{n-1} b)_\circ$.

Moreover, if u_1^{n-1} is a right unit and $u_{n-1} u_1^{n-2}$ is a left unit, then

$$\forall a, b \in A,\ \forall c_1^{n-1} \in A;\ a\rho b \Longleftrightarrow (c_1^{i-1} a c_i^{n-1})_\circ \rho (c_1^{i-1} b c_i^{n-1})_\circ \text{ for } i = 1, \ldots, n \tag{2.4}$$

Indeed, by assumption (2.1) we have

$$(u_{n-2}, a, c_1^{n-2})_\circ \rho (u_{n-2}, b, c_1^{n-2})_\circ$$

and by (the just proved) (2.3) for $i = n$ we have

$$(u_{n-1} u_1^{n-3} (u_{n-2} a c_1^{n-2})_\circ c_{n-1})_\circ \rho (u_{n-1} u_1^{n-3} (u_{n-2} b c_1^{n-2})_\circ c_{n-1})_\circ \Longrightarrow$$

$$\Longrightarrow (a c_1^{n-1})_\circ \rho (b c_1^{n-1})_\circ$$

Moreover, putting the elements $c_1^{n-1} \in A$ in (2.1) we obtain:

$$(u_{n-2} a u_1^{n-2})_\circ \rho (u_{n-2} b u_1^{n-2})_\circ \Longrightarrow$$

$$\Longrightarrow (u_{n-3}(u_{n-2}au_1^{n-2})_\circ u_{n-1}u_1^{n-4})_\circ \rho (u_{n-3}(u_{n-2}au_1^{n-2})_\circ u_{n-1}u_1^{n-4})_\circ \Longrightarrow$$

$$\Longrightarrow (u_{n-3}^{n-2}au_1^{n-4})_\circ \rho (u_{n-3}^{n-2}au_1^{n-4})_\circ \Longrightarrow \ldots \Longrightarrow \ldots \Longrightarrow$$

$$\Longrightarrow (u_{n-1}(u_1^{n-2}au_1)_\circ u_2^{n-1})_\circ \rho (u_{n-1}(u_1^{n-2}bu_1)_\circ u_2^{n-1})_\circ \Longrightarrow a\rho b.$$

Corollary 2.2 *If $(A, ()_\circ)$ is an n-group and ρ an equivalence relation on A satisfying (2.1) then ρ is a congruence relation on $(A, ()_\circ)$.*

We will give in the sequel a property of the congruences on n–semigroups and their binary reduces.

Proposition 2.3. *Let $(A, ()_\circ)$ be an n-semigroup and $u_1^{n-2} \in A$ arbitrary fixed elements. Then the following hold:*

1) If ρ is a congruence relation on the n-semigroup A, then ρ is a congruence on the $red_{u_1^{n-2}}(A, ()_\circ)$;

2) If ρ is a congruence on semigroup $(A, \cdot)$ $\alpha \in End(A, \cdot)$ and $\exists c \in A$ such that (1.3) holds and

$$\forall a, b \in A,\ \forall c_1^{n-1} \in A;\ a\rho b \Longrightarrow \alpha(a)\rho\alpha(b), \tag{2.5}$$

then ρ is a congruence on n-semigroup $ext_{\alpha,c}(A, \cdot)$

The proof follows immediately.

By using Theorem 1.1 above we have too:

Corollary 2.4. *Let $(A, ()_\circ)$ be an n-semigroup with $u_1^{n-2} \in A$ a right unit. If ρ is a congruence on $red_1^{n-2}(A, ()_\circ)$ and the following relation holds*

$$\forall a, b \in A,\ \forall c_1^{n-1} \in A;\ a\rho b \Longrightarrow (u_{n-2}, a, u_1^{n-2})_\circ \rho (u_{n-2}, b, c_1^{n-2})_\circ \tag{2.6}$$

then the ρ is a congruence on $(A, ()_\circ)$ too.

Corollary 2.5. *If $(A, ()_\circ)$ is an n-group and $c \in A$, then ρ is a congruence on the n-group A, if and only if ρ is a congruence on the binary reduce $red_c(A, ()_\circ)$ and*

$$\forall a, b \in A;\ a\rho b \Longrightarrow (c, a, \overline{c}, \overset{n-3}{c})_\circ \rho (c, b, \overline{c}, \overset{n-3}{c})_\circ \tag{2.7}$$

holds.

References

1. **Dörnte, W.,** Untersuchungen über eine verallgemeinerten Gruppenbegriff, Math.Z. 29 (1928), p.1-19

2. **Hosszú, M.,** On the explicit form of n-group operations, Publ.Math.Debrecen, 10, 1963, p.88-92

3. **Pop S. Maria,** Contribuţii la teoria n-semigrupurilor, Teza de doctorat, Univ.Babeş-Bolyai, Cluj-Napoca, 1977

4. **Pop S. Maria,** Congruente in n-semigrupuri, Bul.Ştiinţ.Seria B, vol.VI, Univ. Baia Mare, 1983, p.5-10

5. **Purdea, I.,Pic, Gh.** Tratat de algebră modernă, vol.I, Editura Academiei R.S.R., 1977

6. **Usan J.,** Neutral operation of n-grupoids, Rev of research, Fac. of Sci. Univ of Novi Sad, Math. Ser 18,2(1988), 117-126(in Russian)

7. **Usan J.,** A comment on n-groups, Rev of research, Fac. of Sci. Univ of Novi Sad, Math. Ser 24,1 (1994), 281-288

8. **Usan J.,** On congruences on n-groups, Novi Sad Journal of Math. 27,2(1997), 89-100

9. **Usan J.,** Congruences of n-group and of associated Hosszú-Gluskin algebras, Novi Sad Journal of Math. 28, 2(1998), 91-108

10. **Zupnik, D.,** Polyadic semigroups, Publ.Math., Debrecen, 14, 1967, p.273-279

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