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A PRECONDITIONING METHOD OF ILL CONDITIONED MATRICES USING WAVELET BASES

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Abstract After discretizations with respect to two different wavelet bases of the partial differential equations (PDEs), we obtain a big sparse ill-conditioned linear system of equations. For discretizing of PDEs with wavelet method, this paper presents a preconditioning technique for linear systems involving the operator such that the system becomes a sparse systems in the wavelets basis. In fact the condition number of the matrix involved in the solution of PDEs, after a diagonal preconditioning appears to be bounded. The orthogonal property of the wavelets is used to construct efficient iterative methods for the solution of the resultant linear algebraic systems.

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1. Introduction. The problem to be efficient a preconditioner for linear systems Ax=b, one hopes that: the preconditioner C can be found easily or C can be computed by little computational cost, the linear system Cy=d requires much less operations than Ax=b, and the spectrum of $C^{-1}A$ must be considerably better than that of A. Applying wavelets to discretize differential equation appears to be a very attractive idea. In finite element type methods, piecewise polynomial trial functions may be replaced by wavelets (Wavelet-Galerkin method).

Among the good features of the wavelet method (WM) we have a class of fast algorithms, all based on the fast wavelet transform, such as the fast matrix-vector multiplication.

2. Wavelet bases. Let be the partial differential equation

(1) $$Lu = f.$$

We introduce a function $\varphi(x) \in L^2(R)$ called the father wavelet (or scaling function), with a compact support [0,a], a > 0; see [4]. The function $\varphi(x)$ has the property that

(2) $$\varphi(x-k), \qquad k \in Z$$

form an orthonormal sequence in $L^2(R)$. Let V_o be the closed linear subspace of $L^2(R)$ generated by (2). The multiresolution analysis (MRA), depending on this $\varphi(x)$, is given as follows:

(i) $f(x) \in V_o$ if only if $F(2^j x) \in V_j$;

(ii) $... \subset V_{j-1} \subset V_j \subseteq V_{j+1} \subseteq ...$;

(iii) $\overline{\cup_{-\infty}^{\infty} V_j} = L^2(R)$ and $\cap_{-\infty}^{\infty} V_j = 0$;

(iv) The sequence (2) forms a Riesz basis of V_o.

Let W_j denote the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$. From MRA (iii), we also have $\oplus_{-\infty}^{\infty} W_j = L^2(R)$. There exists at least one function $\psi(x) \in W_0$ such that

(3) $$\psi(x-k), \qquad k \in Z$$

is an orthonormal basis of W_0 (see [2] and [3]). The $\psi(x)$ is called the mother wavelet. We then construct the following two wavelet sequences:

(4) $$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k), \qquad j,k \in Z,$$

and

(5) $$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \qquad j,k \in Z.$$

The wavelet sequence $\{\psi_{j,k}(x)\}$ forms a Riesz basis of Sobolev spaces $H^s(R)$ for $s \geq 0$ (see [3]).

The operator L from (1) can be projected on the subspace V_J or on the subspace W_J (J is fixed) with respect to base B_1 respectively to base B_2:

(6) $$B_1 = (\varphi_{j,k}(x))_k, \qquad k \in Z,$$

(7) $$B_2 = \bigcup_{-\infty < j \leq J-1} (\psi_{j,k}(x))_k, \qquad k \in Z.$$

Here B_1 comes from the father wavelet $\varphi(x)$ and B_2 comes from the mother wavelet $\psi(x)$.

Another wavelet base can be constructed on the use of a function θ, obtained as an autocorrelation function of a compactly scaling function of the type we justdescribed above, see [5,6,7]. This function θ is:

(8) $$\theta(x) := (\varphi * \varphi(-\cdot))(x)$$

where * denotes the convolution product, and for integer value of x this becomes

$$\theta(k) = \int \varphi(k)\varphi(x-k)dx \tag{9}$$

We will indicate by U_j the linear span of the set $\{\theta(2^j x - k), k \in\}$. It is possible to prove that $\{U_j\}$ forms a multiresolution analysis where θ plays the role of (nonorthogonal) scaling function. The set $\{2^{j/2}\theta(2^j x - k), k \in\}$ is a Riesz's basis for U_j.

3. Preconditioning techniques. Suppose that L from (1) is a symmetric and coercive operator. Let $u_J = \sum u_{jk}\psi_{jk}$ be the Galerkin projection of u in B_2, the coefficients u_{jk} are then determined by following linear system of equations

$$Au = f. \tag{10}$$

The matrix A is sparse, let P be the permutation matrix which relabels ψ_{jk} in such way that

$$A = P\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}P^T. \tag{11}$$

Let N=dim B_1, then B is a full square matrix with order O(logN). As a result, the eigenvalues of A are mostly 1, with a few exceptions caused by the wavelets near the boundary.

The further improve the performance of the conjugate gradient method, we have a natural choice of preconditioner for matrix A. Let B be the matrix in (11). since B is small, it is easy to invert B by a direct method. Evidently B and B^{-1} are symmetric and positive definite, hence there exists Cholesky decomposition, say $B^{-1} = LL^T$. Then SS^T is a good preconditioner where

$$S = P\begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix}. \tag{12}$$

That is instead of solving Ax=b, we solve the equivalent problem

$$S^T ASy = S^T, \quad x = Sy, \tag{13}$$

where $S^T AS$ is a well-conditioned matrix., and the condition number is uniformly bounded by a constant which does not depend on j, the action of P can be easily derived during the computation, so we only have to compute and store the lower triangular matrix L which has $O(\log^2 N)$ entries. Theoretically the condition number of $S^T AS$ is one. The preconditioner S is made possible by the orthogonality property of wavelets, see [8,9].

Alternatively, one could think of using other preconditioning techniques based not on the MRA U_j but on the interpellating MRA U_j itself. In fact, following D. Donoho [7], one may introduce the space $U_j \subset U_j$ defined by

$$(14) \qquad U_j = span\{\theta_{j+1,2k+1}, k \in Z\}$$

The change of basis of the spaces U_j and $\bar{U}_j$ is performed by means of the fast interpolating wavelet transform,which has the same algorithmical structure of the usual fast wavelet transform. Rescaling the matrix arising from Wavelet Galerkin method, by a diagonal matrix P one obtains a matrix whose condition number satisfies only

$$(15) \qquad cond(PAP) \leq C2^j.$$

Such technique, though much less effective, may be still of interest, in view of an application, where the number of unknowns is reduced by considering only those degrees of freedom which are relevant to the problem.

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