Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică - Informatică, Vol. XVII(2001), Nr. 1-2, 117-126

# DIFFERENTIAL OPERATORS ON BUNDLE OF ACCELERATIONS IN HOMOGENEOUS APPROACH

**Adrian SANDOVICI**

**Abstract.** Using a special $(h, v_1, v_2)$ metrical structure on the bundle of accelerations, the divergence of a field, the gradient and the Laplacean of a function on $Osc^{(2)}M$ are defined in this paper. In the case of homogeneous lift of a Riemannian metric to bundle of accelerations we obtain some results on differential operators.

**MSC:** 53C60

**Keywords and phrases:** bundle of accelerations, $(\alpha, \beta, \eta)$ Sasaki lift to bundle of accelerations of a Riemannian structure, homogeneous Sasaki lift, differential operator.

## 1. Introduction

Let $(E = Osc^{(2)}M, \pi^2, M)$ be the 2-osculator bundle of a real, smooth, n-dimensional manifold $M$. For a local chart $(U, \varphi = (x^i))$ on $M$, its induced local chart on $E = Osc^{(2)}M$ will be denoted by $\left(\left(\pi^2\right)^{-1}(U), \Phi = \left(x, y^{(1)i}, y^{(2)i}\right)\right)$.

Let $R^{(n)} = (M, \gamma)$ be a Riemannian space, $M$ being a real n-dimensional differentiable manifold and $\gamma$ a Riemannian metric on M, having the local coordinates $\gamma_{ij}(x)$, $x \in U \subset M$. We can extend $\gamma_{ij}$ to $\pi^{-1}(U) \subset E$, setting:

$$\left(\gamma_{ij} \circ \pi\right)(u) = \gamma_{ij}(x), u \in \pi^{-1}(U), \pi(u) = x \tag{1.1.}$$

In this case $\gamma_{ij} \circ \pi$ gives a d-tensor field on E. We also denoted it by $\gamma_{ij}$. If we denote by $\gamma^i_{jk}$ the Christoffel symbols of $\gamma$ we have the following result:

**Theorem 1.1.** *The coefficients of the nonlinear connection on $\widetilde{E}$ determined only by the Riemannian structure $\gamma(x)$ are given by:*

$$\begin{cases} \underset{(1)}{N}{}^i_j\,(x,y^{(1)}) = \gamma^i_{j0} \\ \underset{(2)}{N}{}^i_j\,(x,y^{(1)},y^{(2)}) = \frac{1}{2}A^i_j + \gamma^i_{j\bar{0}} \end{cases} \tag{1.2.}$$

*where "0" means the contraction by* $(y^{(1)})$, *"*$\bar{0}$*" means the contraction by* $(y^{(2)})$ *and* $A^i_j = \left(\frac{\partial \gamma^i_{j0}}{\partial x^p} y^{(1)p} - \gamma^i_{0m}\cdot\gamma^m_{j0}\right)$.

The nonlinear connection N on $E$ provides the existence of the adapted basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$ of the tangent space $T_uE$. These vector fields are given by the following relation:

$$\begin{cases} \delta_k = \frac{\partial}{\partial x^k} - \underset{(1)}{N}{}^i_k \cdot \frac{\partial}{\partial y^{(1)i}} - \underset{(2)}{N}{}^i_k \cdot \frac{\partial}{\partial y^{(2)i}} \\ \overset{(1)}{\delta}_k = \frac{\partial}{\partial y^{(1)k}} - \underset{(1)}{N}{}^i_k \cdot \frac{\partial}{\partial y^{(2)i}} \\ \overset{(2)}{\delta}_k = \frac{\partial}{\partial y^{(2)k}} \end{cases} \tag{1.3.}$$

Further, we need the following result with respect to the vector fields of the adapted basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$:

**Theorem 1.2.** *In the case of the nonlinear connection on* $\widetilde{E}$ *determined only by the Riemannian structure* $\gamma(x)$, *the local coefficients of Lie brackets of the vector fields of the adapted basis* $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$ *are given by:*

$$\begin{cases} \underset{(01)}{R}{}^i_{jk} = r^i_{0jk} \quad \underset{(12)}{R}{}^i_{jk} = 0 \end{cases} \tag{1.4.}$$

$$\begin{cases} \underset{(02)}{R}{}^i_{jk} = \frac{1}{2}\frac{\partial r^i_{0jk}}{\partial x^p}\cdot y^{(1)p} + \\ \frac{1}{2}\cdot\left(\gamma^i_{0m}\cdot r^m_{0jk} + \gamma^m_{0j}\cdot r^i_{0km} + \gamma^m_{0k}\cdot r^i_{0mj}\right) + r^i_{\bar{0}jk} \end{cases} \tag{1.5.}$$

$$\begin{cases} \underset{(11)}{B}{}^i_{jk} = \gamma^i_{jk} & \underset{(22)}{B}{}^i_{jk} = \gamma^i_{jk} \\ \underset{(12)}{B}{}^i_{jk} = \frac{1}{2}\cdot r^i_{0jk} + \gamma^i_{0m}\cdot\gamma^m_{jk} & \underset{(21)}{B}{}^i_{jk} = 0 \end{cases}, \tag{1.6.}$$

*where $r^i_{hjk}(x)$ are the local coefficients of the curvature tensor field of the Riemannian space $(M, \gamma_{ij}(x))$.*

**Theorem 1.3.** ([6]) *The pair* $\Pr ol^2 R^{(n)} = (\overset{\sim}{Osc}^{(2)} M, G)$, *where:*

$$G = \gamma_{ij}(x) \cdot dx^i \otimes dx^j + \gamma_{ij}(x) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + \gamma_{ij}(x) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \quad (1.7.)$$

*is a Riemannian space of dimension 3n, whose metric structure G depends only on the structure $\gamma(x)$ of the apriori given Riemann space $R^{(n)} = (M, \gamma)$.*

We say that G is Sasaki N – lift of the Riemannian structure $\gamma$.

Next, let us consider the homothety $h_t : (x, y^{(1)}, y^{(2)}) \to (x, ty^{(1)}, t^2 y^{(2)})$, $t \in R^*$ on the fibres of fibred bundle $Osc^{(2)} M$. We notice that G is transformed as follows:

$$G \circ h_t (x, y^{(1)}, y^{(2)}) = \gamma_{ij}(x) \cdot dx^i \otimes dx^j + t^2 \cdot \gamma_{ij}(x) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + t^4 \cdot \gamma_{ij}(x) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \quad (1.8.)$$

Therefore we have:

**Theorem 1.4.** *The Sasaki lift G is not homogeneous on the fibres of fibres bundles $Osc^{(2)} M$.*

## 2.The $(\alpha, \beta, \eta)$ Sasaki lift to bundle of accelerations of a Riemannian metric

**Definition 2.1.** *We call the $(\alpha\beta\eta)$ lift to $\overset{\sim}{Osc}^{(2)} M$ of the fundamental tensor field $\gamma_{ij}$ of a Riemannian space $R^{(n)}$ the following tensor field on $\overset{\sim}{Osc}^{(2)} M$ :*

$$\overset{(\alpha\beta\eta)}{G} = g_{ij}(x, y^{(1)}) \cdot dx^i \otimes dx^j + h_{ij}(x, y^{(1)}) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + l_{ij}(x, y^{(1)}) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \quad (2.1.)$$

*where we put:*

$$\begin{cases} g_{ij}(x, y^{(1)}) = \alpha\,(F^2) \cdot \gamma_{ij}(x) \\ h_{ij}(x, y^{(1)}) = \beta\,(F^2) \cdot \gamma_{ij}(x) \\ l_{ij}(x, y^{(1)}) = \eta\,(F^2) \cdot \gamma_{ij}(x) \end{cases} \tag{2.2.}$$

*where $F$ is the Finsler function of Riemannian space $R^{(n)}$ : $F^2 = \gamma_{ij}(x) \cdot y^{(1)i} \cdot y^{(1)j}$, $y_i^{(1)} = \gamma_{ij}(x) \cdot y^{(1)j}$ and $\alpha$, $\beta$, $\eta : R_+ \to R_+$ are differentiable functions.*

**Theorem 2.1.** *The following properties hold:*

*a) The pair $(\widetilde{TOsc}^{(2)} M, \overset{(\alpha\beta\eta)}{G})$ is a Riemannian space;*

*b) $\overset{(\alpha\beta\eta)}{G}$ depends only on the Riemannian metric $\gamma$ of the Riemann space $R^{(n)}$;*

*c) The distributions $N$, $V_1$, $V_2$ are orthogonal with respect to $\overset{(\alpha\beta\eta)}{G}$.*

**Remarks:**

a) For $\alpha = \beta = \eta = 1$ we obtain the Sasaki lift introduced by R. Miron in the monograph [2].

b) For $\alpha(t) = 1$, $\beta(t) = \frac{a^2}{t}$, $\eta(t) = \frac{a^4}{t^2}$ $(a > 0)$ we obtain a new lift called homogeneous Sasaki lift, which will be denoted with $\overset{(0)}{G}$. Therefore we have:

$$\overset{(0)}{G} = \gamma_{ij}(x) \cdot dx^i \otimes dx^j + \tfrac{a^2}{F^2} \cdot \gamma_{ij}(x) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + \tfrac{a^4}{F^4} \cdot \gamma_{ij}(x) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \tag{2.3.}$$

**Definition 2.2.** *A linear connection $D$ on $\widetilde{Osc}^{(2)} M$ is called a $(\alpha\beta\eta)$ metrical connection if $D\,\overset{(\alpha\beta\eta)}{G} = 0$ and $D$ preserves by parallelism the horizontal distribution $N$.*

In the adapted basis, any linear connection on E can be represented in the following way:

$$\begin{cases} D_{\delta_k}\delta_j = \overset{(H)}{L}{}^i_{jk} \cdot \delta_i + \overset{(1)}{L}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(2)}{L}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ D_{\delta_k}\overset{(1)}{\delta}_j = \overset{(3)}{L}{}^i_{jk} \cdot \delta_i + \overset{(V_1)}{L}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(4)}{L}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ D_{\delta_k}\overset{(2)}{\delta}_j = \overset{(5)}{L}{}^i_{jk} \cdot \delta_i + \overset{(6)}{L}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(V_2)}{L}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \end{cases} \tag{2.4.$_1$}$$

$$\begin{cases} D_{\underset{\delta_k}{(1)}} \delta_j = \overset{(H)}{F}{}^i_{jk} \cdot \delta_i + \overset{(1)}{F}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(2)}{F}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ D_{\underset{\delta_k}{(1)}} \overset{(1)}{\delta}_j = \overset{(3)}{F}{}^i_{jk} \cdot \delta_i + \overset{(V_1)}{F}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(4)}{F}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ D_{\underset{\delta_k}{(1)}} \overset{(2)}{\delta}_j = \overset{(5)}{F}{}^i_{jk} \cdot \delta_i + \overset{(6)}{F}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(V_2)}{F}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \end{cases} \qquad (2.4._2)$$

$$\begin{cases} D_{\underset{\delta_k}{(2)}} \delta_j = \overset{(H)}{C}{}^i_{jk} \cdot \delta_i + \overset{(1)}{C}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(2)}{C}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ D_{\underset{\delta_k}{(2)}} \overset{(1)}{\delta}_j = \overset{(3)}{C}{}^i_{jk} \cdot \delta_i + \overset{(V_1)}{C}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(4)}{C}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ D_{\underset{\delta_k}{(2)}} \overset{(2)}{\delta}_j = \overset{(5)}{C}{}^i_{jk} \cdot \delta_i + \overset{(6)}{C}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(V_2)}{C}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \end{cases} \qquad (2.4._3)$$

The systems of functions $( \overset{(H)}{L}{}^i_{jk}, \ldots, \overset{(V_2)}{C}{}^i_{jk} )$ are the coefficients of $D$. We can prove the following result:

**Theorem 3.3.** *There exist the $(\alpha\beta\eta)$ metrical connections on $\widetilde{OSC}^{(2)} M$, which depend on the metric tensor $\gamma$.*

*One of them has the following coefficients:*

$$\overset{(1)}{L}{}^i_{jk} = \overset{(2)}{L}{}^i_{jk} = \overset{(3)}{L}{}^i_{jk} = \overset{(4)}{L}{}^i_{jk} = \overset{(5)}{L}{}^i_{jk} = \overset{(6)}{L}{}^i_{jk} = 0 \qquad (2.5.)$$

$$\overset{(1)}{F}{}^i_{jk} = \overset{(2)}{F}{}^i_{jk} = \overset{(3)}{F}{}^i_{jk} = \overset{(4)}{F}{}^i_{jk} = \overset{(5)}{F}{}^i_{jk} = \overset{(6)}{F}{}^i_{jk} = 0 \qquad (2.6.)$$

$$\overset{(H)}{C}{}^i_{jk} = \overset{(1)}{C}{}^i_{jk} = \overset{(2)}{C}{}^i_{jk} = \overset{(3)}{C}{}^i_{jk} = \overset{(4)}{C}{}^i_{jk} = \overset{(5)}{C}{}^i_{jk} = \overset{(6)}{C}{}^i_{jk} = \overset{(V_1)}{C}{}^i_{jk} = \overset{(V_2)}{C}{}^i_{jk} = 0 \qquad (2.7.)$$

$$\overset{(H)}{L}{}^i_{jk} = \overset{(V_1)}{L}{}^i_{jk} = \overset{(V_2)}{L}{}^i_{jk} = \gamma^i_{jk} \qquad (2.8.)$$

$$\overset{(H)}{F}{}^{i}_{jk} = \frac{\alpha'}{\alpha} \cdot \Lambda^i_{jk} \quad , \quad \overset{(V_1)}{F}{}^{i}_{jk} = \frac{\beta'}{\beta} \cdot \Lambda^i_{jk} \quad , \quad \overset{(V_2)}{F}{}^{i}_{jk} = \frac{\eta'}{\eta} \cdot \Lambda^i_{jk} \tag{2.9.}$$

*where:*

$$\Lambda^i_{jk} = \delta^i_j y^{(1)}_k + \delta^i_k y^{(1)}_j - \gamma_{jk} y^{(1)i} \tag{2.10.}$$

The simplicity of this metrical N-connection and the fact that it is determined only by the Riemannian metric $\gamma$, allows to call it the *canonical* $(\alpha\beta\eta)$ *metrical connection* of the space $(\widetilde{TOsc}^{(2)} M, \overset{(\alpha\beta\eta)}{G})$. In the adapted basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$ the torsion tensor field of the canonical $(\alpha\beta\eta)$ metrical connection of the space $(\widetilde{TOsc}^{(2)} M, \overset{(\alpha\beta\eta)}{G})$ satisfies the following relations:

$$\begin{cases} T(\delta_k, \delta_j) = \overset{(1)}{T}{}^i_{jk} \cdot \delta_i + \overset{(2)}{T}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(3)}{T}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ T\left(\overset{(1)}{\delta}_k, \delta_j\right) = \overset{(1)}{P}{}^i_{jk} \cdot \delta_i + \overset{(2)}{P}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(3)}{P}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ T\left(\overset{(2)}{\delta}_k, \delta_j\right) = \overset{(1)}{Q}{}^i_{jk} \cdot \delta_i + \overset{(2)}{Q}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(3)}{Q}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ T\left(\overset{(1)}{\delta}_k, \overset{(1)}{\delta}_j\right) = \overset{(1)}{S}{}^i_{jk} \cdot \delta_i + \overset{(2)}{S}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(3)}{S}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ T\left(\overset{(2)}{\delta}_k, \overset{(1)}{\delta}_j\right) = \overset{(1)}{V}{}^i_{jk} \cdot \delta_i + \overset{(2)}{V}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(3)}{V}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \\ T\left(\overset{(2)}{\delta}_k, \overset{(2)}{\delta}_j\right) = \overset{(1)}{U}{}^i_{jk} \cdot \delta_i + \overset{(2)}{U}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(3)}{U}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \end{cases} \tag{2.11.}$$

**Theorem 2.4.** *The local components of the torsion tensor field of the canonical* $(\alpha\beta\eta)$ *metrical connection are given by:*

$$\overset{(1)}{T}{}^i_{jk} = \overset{(2)}{P}{}^i_{jk} = \overset{(1)}{Q}{}^i_{jk} = \overset{(2)}{Q}{}^i_{jk} = \overset{(3)}{Q}{}^i_{jk} = 0 \tag{2.12.}$$

$$\overset{(1)}{S}{}^i_{jk} = \overset{(2)}{S}{}^i_{jk} = \overset{(3)}{S}{}^i_{jk} = \overset{(1)}{V}{}^i_{jk} = \overset{(2)}{V}{}^i_{jk} = \overset{(1)}{U}{}^i_{jk} = \overset{(2)}{U}{}^i_{jk} = \overset{(3)}{U}{}^i_{jk} = 0 \tag{2.13.}$$

$$\overset{(2)}{T}{}^{i}_{jk} = \underset{(01)}{R}{}^{i}_{jk}, \quad \overset{(3)}{T}{}^{i}_{jk} = \underset{(02)}{R}{}^{i}_{jk}, \tag{2.14.}$$

$$\overset{(1)}{P}{}^{i}_{jk} = \overset{(H)}{F}{}^{i}_{jk}, \quad \overset{(3)}{P}{}^{i}_{jk} = \underset{(12)}{B}{}^{i}_{jk}, \quad \overset{(3)}{V}{}^{i}_{jk} = -\overset{(V_1)}{F}{}^{i}_{jk} \tag{2.15.}$$

## 3. Differential operators

Let $F(\breve{E})$ be the ring of the smooth functions on $\breve{E}$ and let $X(\breve{E})$ be the $F(\breve{E})-$ module of vector fields on $\breve{E}$. We can write:

$$X = hX + v_1X + v_2X \tag{3.1.}$$

for any $X \in X(\breve{E})$. It corresponds to the possibility to write any $X \in X(\breve{E})$ in the following form:

$$X = \overset{(h)}{X}{}^{i}(u) \cdot \delta_i + \overset{(v_1)}{X}{}^{i}(u) \cdot \overset{(1)}{\delta}_i + \overset{(v_2)}{X}{}^{i}(u) \cdot \overset{(2)}{\delta}_i, \quad u = (x, y^{(1)}, y^{(2)}) \tag{3.2.}$$

We suppose that the manifold $M$ is a paracompact one. In this case $\breve{E}$ is an orientable Riemannian manifold with respect to Riemannian metric $\overset{(\alpha\beta\eta)}{G}$. Thus, general considerations on divergence, gradient and Laplacean done in [5] applies to $\breve{E}$. Let us denote by $L_X$ the Lie derivative with respect to $X \in X(\breve{E})$ and by $dV$ the $3n-$ form on $Osc^2M$. Then the divergence of $X \in X(\breve{E})$ is defined by the equation:

$$L_X dV = (divX)dV \tag{3.3.}$$

Let $D$ be a $(\alpha\beta\eta)$ metrical connection. Following the ideas from [1] we obtain:

$$divX = Trace\,(Y \to D_Y X + T(X, Y))$$

If we look for the matrix of the linear operator $Y \to D_Y X + T(X, Y)$ in the adapted basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$ then we obtain:

**Theorem 3.1.** *For any* $X = \overset{(h)}{X}{}^i \cdot \delta_i + \overset{(v_1)}{X}{}^i \cdot \overset{(1)}{\delta}_i + \overset{(v_2)}{X}{}^i \cdot \overset{(2)}{\delta}_i \in X(\tilde{E})$ *we have the following formula:*

$$divX = div_h X + div_{v_1} X + div_{v_2} X \tag{3.4.}$$

*where:*

$$\begin{aligned} div_h X &= \overset{(h)}{X}{}^i{}_{|i} + \overset{(h)}{X}{}^k \left( \overset{(1)}{T}{}^i{}_{ik} - \overset{(2)}{P}{}^i{}_{ki} - \overset{(3)}{Q}{}^i{}_{ki} \right) \\ div_{v_1} X &= \overset{(v_1)}{X}{}^i{}_{|i} + \overset{(v_1)}{X}{}^k \left( \overset{(1)}{P}{}^i{}_{ik} + \overset{(2)}{S}{}^i{}_{ik} - \overset{(3)}{V}{}^i{}_{ki} \right) \\ div_{v_2} X &= \overset{(v_2)}{X}{}^i{}_{||i} + \overset{(v_2)}{X}{}^k \left( \overset{(1)}{Q}{}^i{}_{ik} + \overset{(2)}{V}{}^i{}_{ik} + \overset{(3)}{U}{}^i{}_{ik} \right) \end{aligned} \tag{3.5.}$$

The natural splitting $TOsc^2M = NOsc^2M \oplus V_1Osc^2M \oplus V_2Osc^2M$ produced by $N$ determines the tree partial gradient operators:

$$\begin{aligned} grad_h f &= g^{ij}(\delta_i f) \cdot \delta_j \\ grad_{v_1} f &= h^{ij}(\overset{(1)}{\delta}_i f) \cdot \overset{(1)}{\delta}_j \\ grad_{v_2} f &= l^{ij}(\overset{(2)}{\delta}_i f) \cdot \overset{(2)}{\delta}_j \end{aligned} \tag{3.6.}$$

and hence we state:

**Definition 3.1.** *Let* $f \in F(\tilde{E})$. *The vector field*

$$gradf = grad_h f + grad_{v_1} f + grad_{v_2} f \tag{3.7.}$$

*is called the global gradient with respect to* $\overset{(\alpha\beta\eta)}{G}$.

**Definition 3.2.** *The scalar field* $\Delta f = div(gradf)$ *is called the Laplacean of the function* $f$ *with respect to the metrical structure* $\overset{(\alpha\beta\eta)}{G}$ *and the* $(\alpha\beta\eta)$ *metrical connection* $D$.

For a metrical connection D we can write:

$$\Delta f = \Delta_h f + \Delta_{V_1} f + \Delta_{V_2} f \tag{3.8.}$$

where:

$$\Delta_h f = \left(g^{ij}(\delta_j f)\right)_{|i} + \left(g^{kp}(\delta_p f)\right) \cdot \left(\overset{(1)}{T}{}^i_{ik} - \overset{(2)}{P}{}^i_{ki} - \overset{(3)}{Q}{}^i_{ki}\right)$$

$$\Delta_{v_1} f = \left(h^{ij}(\overset{(1)}{\delta}_j f)\right)_{||i} + \left(h^{kp}(\overset{(1)}{\delta}_p f)\right) \cdot \left(\overset{(1)}{P}{}^i_{ik} + \overset{(2)}{S}{}^i_{ik} - \overset{(3)}{V}{}^i_{ki}\right) \quad (3.9.)$$

$$\Delta_{v_2} f = \left(l^{ij}(\overset{(2)}{\delta}_j f)\right)_{|||i} + \left(l^{kp}(\overset{(2)}{\delta}_p f)\right) \cdot \left(\overset{(1)}{Q}{}^i_{ik} + \overset{(2)}{V}{}^i_{ik} + \overset{(3)}{U}{}^i_{ik}\right)$$

**Particular cases:**

**A**. Let $\overset{(\alpha\beta\eta)}{G}$ be the metrical structure given by (2.1.) and $D$ be the canonical $(\alpha\beta\eta)$ metrical connection given by (2.5.) – (2.10.). In this case we have:

**Theorem 3.2.** *The following relations hold:*

$$\begin{cases} div\left(\overset{(1)}{\Gamma}\right) = 0\,, \quad div\left(\overset{(2)}{\Gamma}\right) = n\left[1 + \left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} + \frac{\eta'}{\eta}\right) \cdot F^2\right] \\ \Delta \overset{(1)}{E} = \frac{2n}{\beta}\left[1 + \left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} + \frac{\eta'}{\eta}\right) \cdot F^2\right] - \frac{4F^2 \cdot \beta'}{\beta^2} \end{cases} \quad (3.10.)$$

*where* $\overset{(1)}{\Gamma}$ *and* $\overset{(2)}{\Gamma}$ *are Liouville vector fields given by:*

$$\overset{(1)}{\Gamma} = y^{(1)i} \cdot \frac{\partial}{\partial y^{(2)i}}, \qquad \overset{(2)}{\Gamma} = y^{(1)i} \cdot \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \cdot \frac{\partial}{\partial y^{(2)i}} \quad (3.11.)$$

*and:*

$$\overset{(\Phi)}{E} = \Phi \cdot F^2, \quad \Phi = \Phi(F^2) \quad (3.12.)$$

*is so called the* $\Phi-$*energy of first order.*

**B.** If we consider $\overset{(0)}{G}$ the homogeneous lift (2.3.) and $D$ the corresponding canonical metrical connection, then we obtain:

**Theorem 3.3.**

*a) The following relations hold:*

$$\begin{cases} div\left(\overset{(1)}{\Gamma}\right) = 0\,, \quad div\left(\overset{(2)}{\Gamma}\right) = -2n \\ \Delta \overset{(\Phi)}{E} = 4F^2 \cdot [\Phi'' \cdot F^4 + (3-n) \cdot F^2 \cdot \Phi' + (1-n) \cdot \Phi] \end{cases} \quad (3.13.)$$

*b)* $\overset{(\Phi)}{E}$ *is harmonic if and only if:*

$$\Phi(t) = c_1 \cdot \frac{1}{t} + c_2 \cdot t^{n-1}, \quad c_1, c_2 \in R \tag{3.14.}$$

**Aknowledgements.** A version of this paper was presented at the Centennial Gh. Vrănceanu, $1^{st}-4^{th}$ July, Romanian Academy and University of Bucharest, Romania.

**References**

[1] ANASTASIEI, M., *On Divergence in Finsler Spaces*, Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1994, 1–8.

[2] MIRON, R., *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics*, Kluwer Academic Publishers, 1997.

[3] MIRON, R., *The Homogeneous Lift of a Riemannian Metric*, An. St. Univ. "Al. I. Cuza", Iasi (to appear).

[4] MIRON, R., AND ANASTASIEI, M., *The Geometry of Lagrange Spaces; Theory and Applications*, Kluwer Academic Publishers, 1994.

[5] SPIVAK, M., *A Comprehensive Introduction to Differential Geometry*, vol. IV, Publish of Perish, USA, 1975.

[6] UDRISTE, C., AND BALAN, V., *Differential Operators and Convexity on Vector Bundles Endowed with* $(h, v)$ *– Metrics*, An. St. Univ. "Al. I. Cuza", Iasi, Tom. XLIII, s. I. a, Matematica, 1991, $f_1$, 37–50.

Received: 01.08.2001

Colegiul Naţional "Petru Rareş"
5600, Piatra Neamţ
E–mail: *adrisand@hotmail.com*