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THE ABERRATION OF LIGHT AND THE SPACE-TIME METRICS REDUCTION IN THE CASE OF ROTATION TO CANONICAL FORM

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Abstract. This paper presents the reduction to canonical form of the space-time metrics corresponding to a rotation of angular speed w round the axis Oz. An interpretation of this is also provided.
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1. Notations

In generalized spherical coordinates:

$$x^0 = t \in \mathbb{R}, \quad x^1 = r \in [0,\infty), \quad x^2 = \theta \in [0,\pi], \quad x^3 = \phi \in \mathbb{R} \tag{1}$$

in geometrized units [1]

$$\begin{cases} t = ct_{ph}, \quad r = r_{ph}, \quad \theta = \theta_{ph}, \quad \phi = \phi_{ph}, \\ m = \dfrac{G_{ph}}{c^2} M_{ph}, Q \dfrac{G^{\frac{1}{2}}}{c^2} Q_{ph}, \omega = \dfrac{d\phi}{dt} = \dfrac{1}{c}\omega_{ph,} \\ v = \dfrac{1}{c} V_{ph}, a = \dfrac{1}{c^2} a_{ph} \end{cases} \tag{2}$$

$$\begin{cases} \text{c = velocity of light, and} \\ \text{G = the gravitational constant} \end{cases} \tag{3}$$

the expression of metrics for the space-time next to a spherical body is under the form

$$ds^2 = g_{00}dx^{0^2} + g_{11}dx^{1^2} + g_{22}dx^{2^2} + g_{33}dx^{3^2} \tag{4}$$

The index "ph" signifies the fact that the respective quantities are measured in physical units, m, Q, ω, v, a denote respectively the mass, electrical charge, anguler speed, linear velocity and the acceleration.

With the Reissner- Nordström metrics, the components of the fundamental metric tensor are:

$$g_{00} = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} > 0; \qquad g_{11} = -\frac{1}{g_{00}} < 0; \tag{5}$$

$$g_{22} = -r^2 < 0; \qquad g_{33} = -r^2 \sin^2\theta < 0$$

2.Determination of metrics

Supposing a point $A(r,\theta,\phi)$ rotates round the axis Oz at an angular speed

$$\omega = \frac{d\phi}{dt}$$

instead of the ϕ coordinate we will have take $\phi + \omega t$.

For point A the following metrics will have to be taken:

$$ds^2 = (g_{00} + \omega^2 g_{33})(dt)^2 + g_{11}(dr)^2 + g_{22}(d\theta)^2 + g_{33}(d\phi)^2 + 2\omega g_{33} d\phi dt \tag{6}$$

Noting with

$$v_0 = dt;\ v_1 = dr;\ v_2 = rd\theta;\ v_3 = r\sin\theta d\phi \tag{7}$$

the components of vector

$$\mathbf{v} = (v_0, v_1, v_2, v_3) \tag{8}$$

from the vector space R^4 metrics (6) becomes the real quadric form

$$ds^2 : R^4 \to R;$$
$$ds^2 = a_{00}v_0^2 + a_{11}v_1^2 + a_{22}v_2^2 + a_{33}v_3^2 + 2a_{03}v_0v_3 \tag{9}$$

where

$$\begin{cases} a_{00} = 1 - \dfrac{2m}{r} + \dfrac{Q^2}{r^2} - \omega^2 r^2 \sin^2\theta \geq 0, \\ a_{11} = -\left(1 - \dfrac{2m}{r} + \dfrac{Q^2}{r^2}\right)^{-1} \leq 0, \\ a_{22} = a_{33} = -1 < 0, \quad a_{03} = -\omega r \sin\theta \leq 0. \end{cases} \tag{10}$$

The quadratic form matrix has the form

$$A = \begin{bmatrix} a_{00} & 0 & 0 & a_{03} \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{03} & 0 & 0 & a_{33} \end{bmatrix} \tag{11}$$

3.Reduction to canonical form

The reduction of the quadratic form (9) to the canonical form is done through the method of eigenvalues and eigenvectors, as in this way we can determine an orthonormed basis corresponding to an isometry (rotation in R^4).

From the characteristic equation

$$p(\lambda)=\left[\lambda^2-(a_{00}-1)\lambda-(a_{00}+a_{03}^2)\right](a_{11}-\lambda)(-1-\lambda) \quad (12)$$

we get the eigenvalues:

$$\lambda_0=\frac{a_{00}-1+\sqrt{\Delta}}{2}>0;\ \lambda_1=a_{11};\ \lambda_2=-1;\ \lambda_3=\frac{a_{00}-1-\sqrt{\Delta}}{2}<0 \quad (13)$$

where

$$\Delta=(a_{00}-1)^2+4(a_{00}+a_{03}^2)>0 \quad (14)$$

Metrics (9) aquires the form

$$ds^2=\lambda_0 v_0'^2+\lambda_1 v_1'^2+\lambda_2 v_2'^2+\lambda_3 v_3'^2 \quad (15)$$

with

$$v_0'=dt';\ v_1'=dr';\ v_2'=r'd\theta';\ v_3'=r'\sin\theta' d\phi' \quad (16)$$

As $\lambda_1=a_{11}$ and $\lambda_2=-1$ it results that

$$v_1'=v_1,\ \text{and}\ v_2'=v_2 \quad (17)$$

Norming the eigenvectors that have been obtained from the characteristic system

$$\begin{cases}(a_{00}-\lambda)x_0 & + & a_{03}x_3 & =0\\ (a_{11}-\lambda)x_1 & & & =0\\ (-1-\lambda)x_2 & & & =0\\ a_{03}-x_0 & + & (-1-\lambda)x_3 & =0\end{cases} \quad (18)$$

we get the vectors of the new basis

$$B'=\{e_0',e_1',e_2',e_3'\}, \quad (19)$$

where

$$\begin{cases}e_0' & = & \left(\frac{\alpha}{\sqrt{1+\alpha}},0,0,\frac{-1}{\sqrt{1+\alpha^2}}\right)\\ e_1' & = & (0,1,0,0)\\ e_2' & = & (0,0,1,0)\\ e_3' & = & \left(\frac{+1}{\sqrt{1+\alpha^2}},0,0,\frac{\alpha}{\sqrt{1+\alpha^2}}\right)\end{cases} \quad (20)$$

with

$$\alpha=-\frac{a_{00}+1+\sqrt{\Delta}}{2a_{03}}>0 \quad (21)$$

The matrix of transition from the canonical basis

$$B=\{e_0=(1,0,0,0),\ e_1=(0,1,0,0,),\ e_2=(0,0,1,0),\ e_3=(0,0,0,1)\} \quad (22)$$

to basis B' is

$$C = \begin{bmatrix} \frac{\alpha}{\sqrt{1+\alpha^2}} & 0 & 0 & \frac{1}{\sqrt{1+\alpha^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{\sqrt{1+\alpha^2}} & 0 & 0 & \frac{\alpha}{\sqrt{1+\alpha^2}} \end{bmatrix} \quad (23)$$

Insomuch

$$\det C = 1 \quad (24)$$

we have a rotation in the space R^4 with the axes oriented e_1 and e_2 remaining unchanged. The rotation angle, γ , is determined from the formulae:

$$\cos\gamma = \frac{\alpha}{\sqrt{1+\alpha^2}};\ \sin\gamma = \frac{1}{\sqrt{1+\alpha^2}} \text{ or } tg\gamma = \frac{1}{\alpha} \quad (25)$$

Inserting α from (21) and (14) in (25) we obtain

$$tg\gamma = \frac{\sqrt{(a_{00}+1)^2 + 4a_{03}^2} - (a_{00}+1)}{-2a_{03}} \quad (26)$$

namely

$$w = \omega r \sin\theta = \frac{w_{ph}}{c} \quad (28)$$

is the velocity of point A in generalized units namely the aberration constant in nonrelativistic case. Taking $m = 0$ and $Q = 0$, we get

$$tg\gamma = \frac{\sqrt{w^4+4} + w^2 - 2}{2w} = \frac{w}{2}\left(1 + \frac{1}{4}w^2 - \frac{1}{64}w^6 + \frac{1}{516}w^{10} \ldots\right) \approx \frac{w}{2} \quad (29)$$

where in the first approximation in the half aberration constant

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