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MANN ITERATION FOR DIRECT PSEUDOCONTRACTIVE MAPS

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Abstract. In this note we introduce a new class of maps. Let X be a real normed space, and $B \subset X$ be a nonempty set. The map $T : B \to B$ is *direct pseudocontractive* if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\|^2 \leq k\|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \forall x, y \in B.$$

For T a direct pseudocontractive map, we prove the convergence of *Mann iteration* to the fixed point of T.

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1. Introduction. Let H be a real Hilbert space, let $B \subset H$ be a nonempty, convex set. Let $T : B \to B$ be a map. Let $x_1 \in B$, be an arbitrary fixed point. We consider the iteration

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n. \tag{1}$$

The sequence $(\alpha_n)_{n\geq1}$ satisfies: $(\alpha_n)_{n\geq1} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. The last relation implies that $\lim_{n\to\infty} \alpha_n = 0$. A prototype for $(\alpha_n)_{n\geq1}$ is $(1/n)_{n\geq1}$.

Definition 1 *The map T is called pseudocontractive if*

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \forall x, y \in B.$$

In [7] we can see an example of a Lipschitz pseudocontractive map with a unique fixed point for which every non trivial Mann sequence fails to converge. The set B is nonempty, convex and compact.

Definition 2 *The map T is called strongly pseudocontractive if there exists $q \in (0,1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + q\|(I-T)x - (I-T)y\|^2, \forall x, y \in B.$$

In [1], [2], [3], [5], [8], [11] the map T is considered strongly pseudocontractive. The sequence $(x_n)_{n\geq 1}$ given by (1) strongly converges to a fixed point of T.

We introduce the following class of maps:

Definition 3 *The map T is called direct pseudocontractive if there exists $k \in (0,1)$ such that*

$$\|Tx - Ty\|^2 \leq k\|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \forall x, y \in B. \qquad (2)$$

The class of direct pseudocontractive maps is nonempty. If T is a contraction, then T is a direct pseudocontractive map.Picard -Banach Theorem can't be used to find the fixed point of a direct pseudocontractive map. Instead, Mann iteration (1) can be successfully used. Our aim is to give a convergence result for (1). We denote by $F(T) := \{x \in B : Tx = x\}$.

Remark 1 *If T is a direct pseudocontractive map and has $F(T) \neq \emptyset$, then T has a unique fixed point.*

Proof. Let $x^*$and y^* be two distinct fixed points. From (2) we have

$$\begin{aligned} \|Tx^* - Ty^*\|^2 &\leq k\|x^* - y^*\|^2, \\ \|x^* - y^*\|^2 &\leq k\|x^* - y^*\|^2, \\ (1-k)\|x^* - y^*\|^2 &\leq 0,\ k \in (0,1). \end{aligned}$$

Hence $x^* = y^*$. Thus $F(T) = \{x^*\}$.□

The following lemma can be found in [10] as Lemma 4. Also, it can be found in [12] as Lemma 1.2, with an other proof. In [1] can be found as Lemma 2, the proof is similar to the proof of Lemma 1 from [8].

Lemma 4 *[1], [10], [12] Let $(a_n)_{n\geq 1}$ be a nonnegative sequence which verifies where $a_{n+1} \leq (1-\lambda_n)a_n + \sigma_n$, $(\lambda_n)_{n\geq 1} \subset (0,1)$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.*

The following result is proved in [4].

Lemma 5 *[4] Let H be a Hilbert space, the following relation is true for all $x, y \in H$,and for all $\lambda \in (0,1)$:*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \qquad (3)$$

2.The main result.

We are able now to give the main result:

Theorem 6 *Let H be a real Hilbert space, let $B \subset H$ be a nonempty, convex, bounded and closed set and let $T : B \to B$ be a continuous, direct pseudo-contractive map, with $F(T) \neq \emptyset$. Then for each x_1 a fixed point in B, the sequence $(x_n)_{n\geq 1}$ given by (1) converges strongly to the unique fixed point of T.*

Proof. Let $x^* \in F(T)$. From remark 2 we know that $F(T) = \{x^*\}$. Using (2) and (3) we get

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= & \ \|(1-\alpha_n)x_n + \alpha_n T x_n - x^*\|^2 \\
= & \ \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T x_n - x^*)\|^2 \\
= & \ (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|T x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T x_n - x_n\|^2 \\
\leq & \ (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k \|x_n - x^*\|^2 + \\
& +\alpha_n \|T x_n - x_n\|^2 - \alpha_n(1-\alpha_n)\|T x_n - x_n\|^2 \\
\leq & \ [1-(1-k)\alpha_n]\|x_n - x^*\|^2 + \alpha_n^2 \|T x_n - x_n\|^2 .
\end{aligned}$$

The sequence $(\|T x_n - x_n\|^2)_{n\geq 1}$ is bounded, because B is bounded. There exists $M > 0$ such that $\|T x_n - x_n\|^2 < M$, for all $n \geq 1$. We denote $a_n := \|x_n - x^*\|^2$, and we get:

$$a_{n+1} \leq [1-(1-k)\alpha_n]a_n + \alpha_n^2 M.$$

Let us denote by

$$\begin{aligned}
\lambda_n & : = (1-k)\alpha_n, \\
\sigma_n & : = \alpha_n^2 M.
\end{aligned}$$

Observe that $\lambda_n = (1-k)\alpha_n \subset (0,1)$, for all $n \geq 1$. We have $\sum_{n=1}^{\infty} \lambda_n = (1-k)\sum_{n=1}^{\infty} \alpha_n = \infty$. The following relation is true

$$\lim_{n\to\infty} \frac{\sigma_n}{\lambda_n} = \lim_{n\to\infty} \frac{\alpha_n^2 M}{(1-k)\alpha_n} = \frac{M}{1-k} \lim_{n\to\infty} \alpha_n = 0.$$

Thus, we have $\sigma_n = o(\lambda_n)$. From Lemma 1 we get $\lim_{n\to\infty} a_n = 0$. Hence $\lim_{n\to\infty} \|x_n - x^*\| = 0$. The proof is complete.□

Using the Schauder fixed point theorem we give the following corollary:

Corollary 7 *Let H be a real Hilbert space, let $B \subset H$ be a nonempty, convex, compact set and let $T : B \to B$ be a continuous, direct pseudocontractive map. Then for each x_1 a fixed point in B, the sequence $(x_n)_{n\geq 1}$ given by (1) converges strongly to the unique fixed point of T.*

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