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SOME EXAMPLES CONCERNING THE EXTENSIONS OF SEMI-LIPSCHITZ FUNCTIONS ON QUASI-METRIC SPACES

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Abstract. In this article we construct some semi-Lipschitz functions defined on a subset of a quasi-metric space, and then, using a result from [2], for each of them we show two extensions which preserve the smallest semi-Lipschitz constant.

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First we recall some definitions and notations (see [1]).

Definition 1. Let X be a nonempty set and $d : X \times X \to [0,\infty)$ a function which satisfies (i) or (i') and satisfies (ii):

(i) $\quad d(x,y) = 0 \Leftrightarrow x = y$, where $x, y \in X$

(i') $\quad d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, where $x, y \in X$

(ii) $\quad d(x,y) < d(x,z) + d(z,y), \forall x, y, z \in X$.

The function f is called a *quasi-metric* on X and the pair (X,d) is called a *quasi-metric space* T_1*- separated* is case of (i), respectively T_0*-separated* in case of (i').

Remark the difference with respect to a metric space: the symmetry condition $d(x,y) = d(y,x), \forall x, y \in X$ is not required.

Definition 2. A function $f : X \to R$ defined on the quasi-metric space (X,d) is called semi-Lipschitz if there exists a number $L \geq 0$ such that $f(x) - f(y) \leq L \cdot d(x,y), \forall x, y \in X$.

Considering $$\|f\|_d = \sup\left\{\frac{(f(x) - f(y)) \vee 0}{d(x,y)} : x, y \in X, d(x,y) > 0\right\},$$

according to [1], for a semi-Lipschitz $f : X \to R$ function we have $\|f\|_d < \infty$ and $L = \|f\|_d$ will be the smallest semi-Lipschitz constant of the function f.

We remark that all semi-Lipschitz functions $f : X \to R$ defined on the quasi-metric space (X, d) are *d-increasing*:

$x, y \in X, d(x, y) = 0 \Rightarrow f(x) - f(y) \le 0.$

For a nonempty subset $Y \subseteq X$, where (X, d) is a quasi-metric space, denote by $SLipY$ the set of all real valued semi-Lipschitz functions $f : Y \to R$, and for a semi-Lipschitz $f : Y \to R$ function consider the set $E_{Y,d}(f) = \{F \in SLipX : F|_Y = f, \|F\|_d = \|f\|_d\}$ of all extensions of f which preserve the smallest semi-Lipschitz constant.

In [2] it was shown that the set $E_{Y,d}(f)$ is not empty, particularly it contains the following two functions:

$$F_1 : X \to R, F_1(x) = \sup_{y \in Y}\{ f(y) - \|f\|_d \cdot d(y, x) \}, \forall x \in X$$

$$F_2 : X \to R, F_2(x) = \inf_{y \in Y}\{ f(y) + \|f\|_d \cdot d(x, y) \}, \forall x \in X.$$

These functions F_1 and F_2 have a significant role, as in [1] it was proved that for arbitrary $F \in E_{Y,d}(f)$ the following inequalities holds:

$F_1(x) \le F(x) \le F_2(x), \forall x \in X.$

Now we consider the subset $Y = \{-1, 0, 1\} \subset Z \subset R$, the quasi-metric spaces $(Z, d), (R, d_1), (R, d_2)$ where

$$d(x,y) = \begin{cases} 0, \text{ if } x = y \\ 0, \text{if } x = 2n, y = 2n-1 \text{ or } x = 2n, y = 2n+1 \, (n \in Z) \, (KHALIMSKY) \\ 1, \text{ otherwise} \end{cases}$$

$$d_1(x,y) = \begin{cases} x - y, \text{ if } x - y \ge 0 \\ 0, \text{ if } x < y \end{cases}$$

$$d_2(x,y) = \begin{cases} x - y, \text{if } x - y \ge 0 \quad (SORGENFREY) \\ 1, \text{ if } x < y \end{cases}$$

and the functions

$$f, g : Y \to R, f(-1) = -\frac{1}{2}, f(0) = -1, f(1) = 1; \; g(-1) = -\frac{1}{2}, g(0) = 0; g(1) = 1.$$

We remark that (Z, d), (R, d_1) are T_0-separated and (R, d_2) is T_1-separated.

Example 1. The function f is semi-Lipschitz on the quasi-metric space (Y,d), with the semi-Lipschitz constant

$$L=\sup\left\{\frac{f(-1)-f(0)}{d(-1,0)},\frac{0}{d(-1,1)},\frac{f(1)-f(-1)}{d(1,-1)},\frac{f(1)-f(0)}{d(1,0)}\right\}=$$

$$=\left\{\frac{1}{2},0,\frac{3}{2},2\right\}=2,$$

because we have also

$f(0)-f(-1)=-\frac{1}{2}<0=2\cdot d(0,-1)$; $f(0)-f(1)=-2<0=2\cdot d(0,1)$ and obviously $f(y)-f(y)=0\leq 2\cdot d(y,y)=0, \forall y\in Y$.

Now we are going to construct the mentioned extremal extensions of the function f:

$$F_1:Z\to R,\ F_1(x)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,x),\ -1-2\cdot d(0,x),\ 1-2\cdot d(1,x)\right\},\ \forall x\in Z$$

and

$$F_2:Z\to R, F_2(x)=\inf\left\{-\frac{1}{2}+2\cdot d(x,-1),-1+2\cdot d(x,0),1+2\cdot d(x,1)\right\},$$

$\forall x\in Z$. By easy computation we get

a) For $x\in\{\ldots,-5,-4,-3\}$

$$F_1(x)=\sup\left\{-\frac{1}{2}-2\cdot 1,\ -1-2\cdot 1,\ 1-2\cdot 1\right\}=\sup\left\{-\frac{5}{2},-3,-1\right\}=-1$$

$$F_2(x)=\inf\left\{-\frac{1}{2}+2\cdot 1,\ -1+2\cdot 1,\ 1+2\cdot 1\right\}=\inf\left\{\frac{3}{2},1,3\right\}=1$$

b) $$F_1(-2)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,-2),\ -1-2\cdot d(0,-2),\ 1-2\cdot d(1,-2)\right\}=$$

$$=\sup\left\{-\frac{1}{2}-2\cdot 1,\ -1-2\cdot 1,\ 1-2\cdot 1\right\}=\sup\left\{-\frac{5}{2},-3,-1\right\}=-1$$

$$F_2(-2)=\inf\left\{-\frac{1}{2}+2\cdot d(-2,-1),\ -1+2\cdot d(-2,0),\ 1+2\cdot d(-2,1)\right\}=$$

$$=\inf\left\{-\frac{1}{2}+2\cdot 0,\ -1+2\cdot 1,\ 1+2\cdot 1\right\}=\inf\left\{-\frac{1}{2},1,3\right\}=-\frac{1}{2}$$

c) $F_1(-1)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,-1),-1-2\cdot d(0,-1),1-2\cdot d(1,-1)\right\}=$

$$=\sup\left\{-\frac{1}{2}-2\cdot 0,-1-2\cdot 0,1-2\cdot 1\right\}=\sup\left\{-\frac{1}{2},-1\right\}=-\frac{1}{2}$$

$$F_2(-1)=\inf\left\{-\frac{1}{2}+2\cdot d(-1,-1),-1+2\cdot d(-1,0),1+2\cdot d(-1,1)\right\}=$$

$$=\inf\left\{-\frac{1}{2}+2\cdot 0,-1+2\cdot 1,1+2\cdot 1\right\}=\inf\left\{-\frac{1}{2},1,3\right\}=-\frac{1}{2}$$

d) $F_1(0)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,0),-1-2\cdot d(0,0),1-2\cdot d(1,0)\right\}=$

$$=\sup\left\{-\frac{1}{2}-2\cdot 1,-1-2\cdot 0,1-2\cdot 1\right\}=\sup\left\{-\frac{5}{2},-1\right\}=-1$$

$$F_2(0)=\inf\left\{-\frac{1}{2}+2\cdot d(0,-1),-1+2\cdot d(0,0),1+2\cdot d(0,1)\right\}=$$

$$=\inf\left\{-\frac{1}{2}+2\cdot 0,-1+2\cdot 0,1+2\cdot 0\right\}=\inf\left\{-\frac{1}{2},-1,1\right\}=-1$$

e) $F_1(1)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,1),-1-2\cdot d(0,1),1-2\cdot d(1,1)\right\}=$

$$=\sup\left\{-\frac{1}{2}-2\cdot 1,-1-2\cdot 0,1-2\cdot 0\right\}=\sup\left\{-\frac{5}{2},-1,1\right\}=1$$

$$F_2(1)=\inf\left\{-\frac{1}{2}+2\cdot d(1,-1),-1+2\cdot d(1,0),1+2\cdot d(1,1)\right\}=$$

$$=\inf\left\{-\frac{1}{2}+2\cdot 1,-1+2\cdot 1,1+2\cdot 0\right\}=\inf\left\{\frac{3}{2},1\right\}=1$$

f) $F_1(2)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,2),-1-2\cdot d(0,2),1-2\cdot d(1,2)\right\}=$

$$=\sup\left\{-\frac{1}{2}-2\cdot 1,-1-2\cdot 1,1-2\cdot 1\right\}=\sup\left\{-\frac{5}{2},-3,-1\right\}=-1$$

$$F_2(2)=\inf\left\{-\frac{1}{2}+2\cdot d(2,-1),-1+2\cdot d(2,0),1+2\cdot d(2,1)\right\}=$$

$$=\inf\left\{-\frac{1}{2}+2\cdot 1,-1+2\cdot 1,1+2\cdot 0\right\}=\inf\left\{\frac{3}{2},1\right\}=1$$

g) For $x\in\{3,4,5,\ldots\}$

$$F_1(x)=\sup\left\{-\frac{1}{2}-2\cdot d(-1,x),-1-2\cdot d(0,x),1-2\cdot d(1,x)\right\}=$$

$$=\sup\left\{-\frac{1}{2}-2\cdot 1,-1-2\cdot 1,1-2\cdot 1\right\}=\sup\left\{-\frac{5}{2},-3,-1\right\}=-1$$

$$F_2(x)=\inf\left\{-\frac{1}{2}+2\cdot d(x,-1),-1+2\cdot d(x,0),1+2\cdot d(x,1)\right\}=$$

$$=\inf\left\{-\frac{1}{2}+2\cdot 1,-1+2\cdot 1,1+2\cdot 1\right\}=\inf\left\{\frac{3}{2},1,3\right\}=1$$

The extremal, semi-Lipschitz constant preserving extensions of the function f on the quasi-metric space (Z,d) are:

$$F_1(x)=\begin{cases}-1, x\in\{\ldots,-4,-3,-2\}\cup\{0\}\cup\{2,3,4,\ldots\}\\ -\frac{1}{2}, x=-1\\ 1, x=1\end{cases}$$

$$F_2(x)=\begin{cases}1, x\in\{\ldots,-5,-4,-3\}\cup\{1,2,3,\cdots\}\\ -\frac{1}{2}, x\in\{-2,-1\}\\ -1, x=0\end{cases}$$

Example 2. The function g is semi-Lipschitz on the quasi-metric space (Y,d_1), with the semi-Lipschitz constant

$$L=\sup\left\{\frac{g(0)-g(-1)}{d_1(0,-1)},\frac{g(1)-g(-1)}{d_1(1,-1)},\frac{g(1)-g(0)}{d_1(1,0)}\right\}=\sup\left\{\frac{1}{2},\frac{3}{4},1\right\}=1$$

since we have also

$$g(-1)-g(0)=-\frac{1}{2}<0=1\cdot d_1(-1,0), g(-1)-g(1)=-\frac{3}{2}<0=1\cdot d_1(-1,1)$$

$$g(0)-g(1)=-1<1\cdot d_1(0,1)$$

and obviously $g(y)-g(y)=0\leq 1\cdot d_1(y,y)=0, \forall y\in Y$.

We are going now to construct the discussed extremal extensions of the function g:

$G_1:R\to R, G_1(x)=\sup\left\{-\frac{1}{2}-d_1(-1,x),0-d_1(0,x),1-d_1(1,x)\right\}, \forall x\in R$ and

$G_2:R\to R, G_2(x)=\inf\left\{-\frac{1}{2}+d_1(x,-1),0+d_1(x,0),1+d_1(x,1)\right\}, \forall x\in R.$

By computation we obtain:

a) For $x \in (-\infty,-1]$:

$$G_1(x) = \sup\left\{-\frac{1}{2}-(-1-x),\ 0-(-x),1-(1-x)\right\} = \sup\left\{\frac{1}{2}+x, x\right\} = x+\frac{1}{2}, \forall x \in (-\infty,-1]$$

$$G_2(x) = \inf\left\{-\frac{1}{2}+0, 0+0, 1+0\right\} = -\frac{1}{2}, \forall x \in (-\infty,-1]$$

b) For $x \in (-1,0]$:

$$G_1(x) = \sup\left\{-\frac{1}{2}-0, 0-(-x), 1-(1-x)\right\} = \sup\left\{-\frac{1}{2}, x\right\} = \begin{cases} -\frac{1}{2}, \forall x \in \left(-1,-\frac{1}{2}\right] \\ x, \forall x \in \left(-\frac{1}{2},0\right] \end{cases}$$

$$G_2(x) = \inf\left\{-\frac{1}{2}+x+1, 0+0, 1+0\right\} = \inf\left\{x+\frac{1}{2}, 0, 1\right\} = \begin{cases} x+\frac{1}{2}, x \in \left(-1,-\frac{1}{2}\right] \\ 0, x \in \left(-\frac{1}{2},0\right] \end{cases}$$

c) For $x \in (0,1]$:

$$G_1(x) = \sup\left\{-\frac{1}{2}-0, 0-0, 1-(1-x)\right\} = \sup\left\{-\frac{1}{2}, 0, x\right\} = x, \forall x \in (0,1]$$

$$G_2(x) = \inf\left\{\frac{1}{2}+x, x, 1\right\} = x, \forall x \in (0,1]$$

d) For $x \in 1,\infty)$:

$$G_1(x) = \sup\left\{-\frac{1}{2}-0, 0-0, 1-0\right\} = \sup\left\{-\frac{1}{2}, 0, 1\right\} = 1, \forall x \in (1,\infty)$$

$$G_2(x) = \inf\left\{x+\tfrac{1}{2}, x, x\right\} = x, \forall x \in (1,\infty)$$

The extremal, semi-Lipschitz constant preserving extensions of the function g on the quasi-metric space (R, d_1) are:

$$G_1(x) = \begin{cases} x+\frac{1}{2}, x \in (-\infty,-1] \\ -\frac{1}{2}, x \in \left(-1,-\frac{1}{2}\right] \\ x, x \in \left(-\frac{1}{2},1\right] \\ 1, x \in (1,\infty) \end{cases}; \qquad G_2(x) = \begin{cases} -\frac{1}{2}, x \in (-\infty,-1] \\ x+\frac{1}{2}, x \in \left(-1,-\frac{1}{2}\right] \\ 0, x \in \left(-\frac{1}{2},0\right] \\ x, x \in (0,\infty) \end{cases}$$

Example 3. The function g is semi-Lipschitz on the quasi-metric space (Y,d_2), with the semi-Lipschitz constant

$$L=\sup\left\{\frac{0}{d_2(-1,0)},\frac{0}{d_2(-1,0)},\frac{g(0)-g(-1)}{d_2(0,-1)},\frac{0}{d_2(0,1)},\frac{g(1)-g(-1)}{d_2(1,-1)},\frac{g(1)-g(0)}{d_2(1,0)}\right\}=$$

$$=\sup\left\{\frac{1}{2},\frac{3}{4},1\right\}=1$$

since obviously $g(y)-g(y)=0\le 1\cdot d_2(y,y)=0,\forall y\in Y$.

We will construct the discussed extremal extensions of the function g:

$G_1:R\to R, G_1(x)=\sup\left\{-\frac{1}{2}-d_2(-1,x),\ 0-d_2(0,x),\ 1-d_2(1,x)\right\},\forall x\in R$ and

$G_2:R\to R,\ G_2(x)=\inf\left\{-\frac{1}{2}+d_2(x,-1),\ 0+d_2(x,0),\ 1+d_2(x,1)\right\},\forall x\in R.$

We obtain:

$$G_1(x)=\sup\left\{-\frac{1}{2}-(-1-x),0-(-x),1-(1-x)\right\}=x+\frac{1}{2},\forall x\in(-\infty,-1)$$

$$G_1(-1)=\sup\left\{-\frac{1}{2},0-1,1-2\right\}=-\frac{1}{2}$$

$$G_1(x)=\sup\left\{-\frac{1}{2}-1,0-(-x),1-(1-x)\right\}=\sup\left\{-\tfrac{3}{2},x\right\}=x,\forall x\in(-1,0)$$

$$G_1(0)=\sup\left\{-\frac{1}{2}-1,0-0,1-1\right\}=0$$

$$G_1(x)=\sup\left\{-\frac{1}{2}-1,0-1,1-(1-x)\right\}=x,\forall x\in(0,1)$$

$$G_1(1)=\sup\left\{-\frac{1}{2}-1,0-1,1-0\right\}=1$$

$$G_1(x)=\sup\left\{-\frac{1}{2}-1,0-1,1-1\right\}=0,\forall x\in(1,\infty)$$

$$G_2(x)=\inf\left\{-\frac{1}{2}+1,0+1,1+1\right\}=\frac{1}{2},\forall x\in(-\infty,-1)$$

$$G_2(-1)=\inf\left\{-\frac{1}{2},1,1\right\}=-\frac{1}{2}$$

$$G_2(x)=\inf\left\{-\frac{1}{2}+x+1,1,2\right\}=x+\frac{1}{2},\forall x\in(-1,0)$$

$$G_2(0)=\inf\left\{-\frac{1}{2}+1,0,2\right\}=0$$

$$G_2(x)=\inf\left\{-\frac{1}{2}+x+1,x,2\right\}=x,\forall x\in(0,1)$$

$$G_2(1)=\inf\left\{-\frac{1}{2}+2,1,1\right\}=1$$

$$G_2(x)=\inf\left\{-\frac{1}{2}+x+1,x,x\right\}=x+\frac{1}{2},\forall x\in(1,\infty)$$

The extremal, semi-Lipschitz constant preserving extensions of the function g on the quasi-metric space (R,d_1) are:

$$G_1(x)=\begin{cases}x+\frac{1}{2}, x\in(-\infty,-1]\\ x, x\in(-1,1]\\ 0, x\in(1,\infty)\end{cases} \quad ; \quad G_2(x)=\begin{cases}\frac{1}{2}, x\in(-\infty,-1)\\ x+\frac{1}{2}, x\in[-1,0)\cup(1,\infty)\\ x, x\in[0,1]\end{cases}$$

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