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ABOUT VISUAL COMPLEX FUNCTIONS

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Let's take our first look at how these new geometries differ from Euclid's. In any triangle (T)

1). (Angle sum of T) = Π

2). Angular excess E(T)=(Angle sum of T) -Π

Euclidian geometry is thus characterized by the vanishing of E(T)

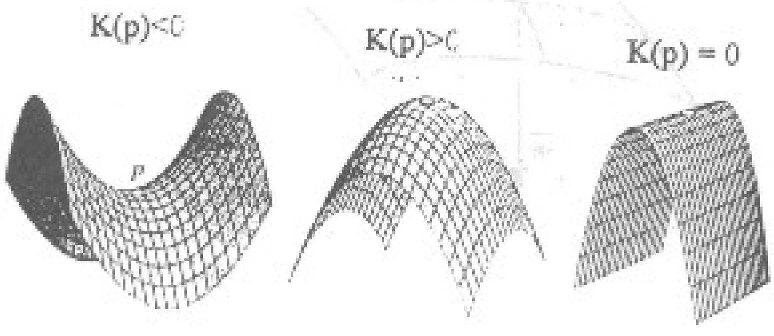
In spherical geometry the angle sum is greater than Π:

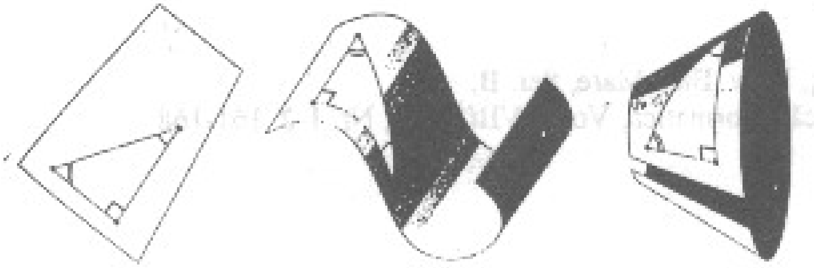
$$E>0 \text{ (Gauss)}$$

In hyperbolic geometry the angle sum is less than Π:

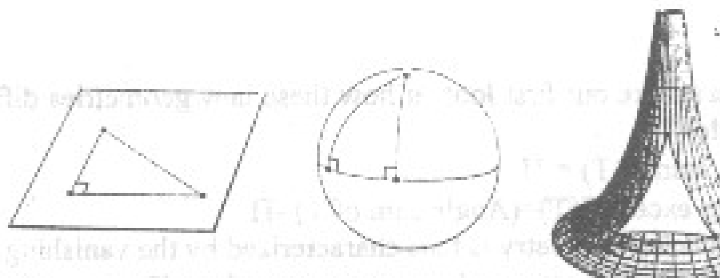
$$E<0 \text{(H. Lambert)}$$

Gauss never published his ideas on non- Euclidian geometry, and the two men who are usually credited for their independent discovery of hyperbolic geometry are Iános Bolyai (1829) and Nikolai Lobachevsky (1832). In 1868 Eugenio Beltrami discovered that hyperbolic geometry could be given a concrete interpretation, via " differential geometry" (the so-called) pseudosphere (figure 1)





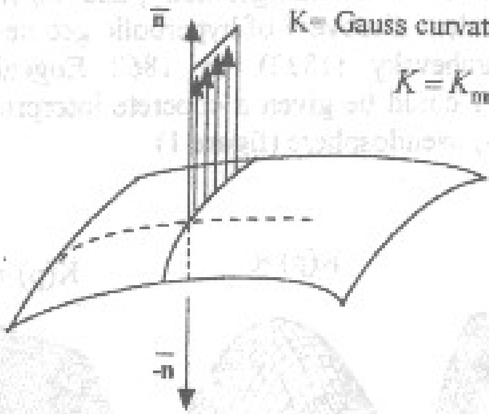
E(T)= 0



Euclidean geometry
K = 0

Spherical geomery
K>0

Hyperbolic geometry
K<0



K= Gauss curvature (total curvature)

$$K = K_{max} \cdot K_{min}$$

Figure (2) shows how we can then define foe example a circle of radius r and centre p. Given three points on the surface we may join them with geodesics to form a triangle of shows tho such triangles T_1 and T_2. Cleary $E(T_1)>0$ like a triangle in spherical geometry, while $E(T_2)< 0$, like a triangle inhyperbolic geometry (figure 2)

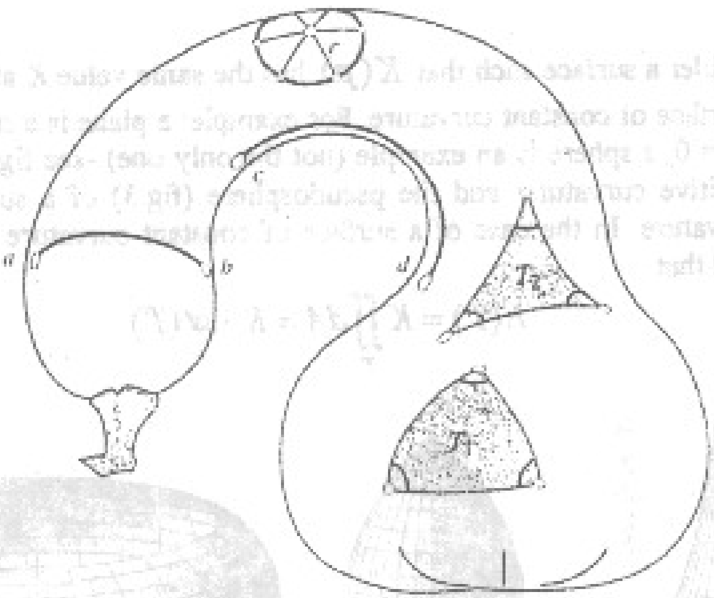


Figure 2

Gaussian curvature

In 1827 Gauss published analysis of the intrinsic and extrinsic geometry of surfaces. He introduced a quantity $K(p)$. This functions $K(p)$ is called the Gaussian curvature. Gauss defined $K(p)$ as follows:

Let (π) be a plane containing the normal vector $\overline{n}$ to the surface at p, and let K be the (signed) curvature at p of the curve in which (π) intersects the surface. The sign of K dependes on whether the centre of curvature is in the direction n or $-n$.

The so-called principal curvatures are the minimum $K_{\min}$ and the maximum $K_{\max}$, values of k as (π) rotates about n. Gauss defined K as the product of the principal curvatures

$$K = K_{\min} \cdot K_{\max} \qquad (3)$$

(see fig.1).

The intrinsic significance of K is exhibited in the following fundamental result: In Δ is an infinitesimal triangle of area dA located at the point p, then

$$E(\Delta) = K(p) \cdot dA \tag{4}$$

Surfaces of Constant Curvature

Consider a surface such that $K(p)$ has the same value K at very point p; we call this a surface of constant curvature. For example: a plane is a surface of constant curvature $K = 0$, a sphere is an example (not the only one) -see fig.4 of a surface of constant positive curvature; and the pseudosphere (fig.3) of a surface of constant negative curvature. In the case of a surface of constant curvature (and only in this case) we find that

$$E(T) = K \iint_T dA = K \cdot \mathscr{A}(T)$$

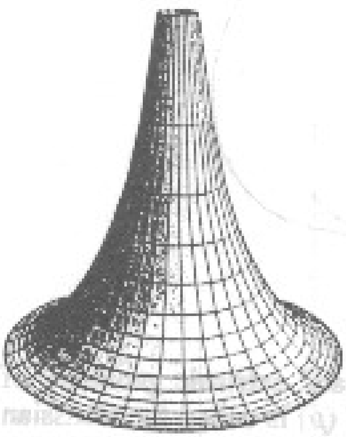


Figure 3

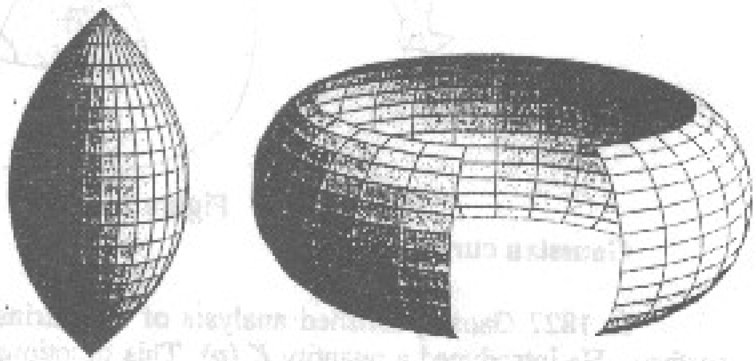


Figure 4

Motions of the plane, sphere and pseudosphere as Möbius transformations

The Euclidian plane is identified with $\mathbb{C}$ its motion are represented by the particularly simple Möbius transformation of the form $M(z) = az + b$. The motions of spherical and hyperbolic geometry are also Möbius transformations!

Stereographic projection onto $\mathbb{C}$ fields a conformal map of the sphere, and the rotations of the sphere thus become complex functions acting on this map. They are the Möbius transformations of the form: $M(z) = \dfrac{az + b}{\bar{b}z + \bar{a}}$ (Gauss, 1819).

Following the same pattern, it is also possible to construct conformal maps (in $\mathbb{C}$) of the pseudosphere thereby transforming its motions into complex functions. The most convenient of these conformal maps is constructed in the unit disc.

The motions of hyperbolic geometry then turn act to be the Möbius automorphisms of this circular map: $M(z) = \dfrac{az + b}{\bar{b}z + \bar{a}}$ (H.Poincaré, 1882).

The fact that a surface has not constant curvature (fig. 2) staps the movement in the complex plane, illustrated by holomorphe functions.

We took the liberty to characterize in the following lines, the non-holomorphe functions from the point of view of some possible geometries.

Naturally, we'll replace $K(p)$, the Gauss curvature with the areolar differential of Pompeiu Dimitrie (1873-1954), knowing the fact that this differential will indicate the distance grade of a function from holomorphism in the point p.

$$\lim_{\delta \to 0} \frac{\oint\limits_{\gamma} f(z)\,dz}{\iint\limits_{D} dx\,dy} = 2i\left(\frac{\partial f}{\partial \bar{z}}\right)_{p}$$

where D - simple conex domain limited by γ - smooth and $p \in D$ (belongs).

The Stokes-Pompeiu formula is

$$2i \iint\limits_{D} \frac{\partial f}{\partial \bar{z}}\,dx\,dy = \oint\limits_{\gamma} f(z)\,dz.$$

We'll define the $E(p)$ ecarte in these geometries with: $E(p)=K(p)\cdot dA$, after the model given by Gauss, but $K(p)=2i\left(\frac{\partial f}{\partial \overline{z}}\right)_p$ and so $E(p)=2i\left(\frac{\partial f}{\partial \overline{z}}\right)_p\cdot dA$.

Obviously if $f(z)\in\mathscr{H}(D)$, $(\forall)\,p\in \overset{\circ}{D}$, than $\frac{\partial f}{\partial \overline{z}}(D)=0, K(p)=0$ and $E(p)=0$. Using this formula we cannot reobtain the cases: $K(p)>0$ or $K(p)<0$.

We'll name the areolar differential $K(p)=2i\left(\frac{\partial f}{\partial \overline{z}}\right)_p$ - complex curvature.

What sort of geometries (surfaces) can generate nonholomorphe function?

Example:

(1) $f(z)=\overline{z}+g(z),\ g(z)\in\mathscr{H}(\mathrm{D}),\ \gamma=F_r D$, smooth

$$\oint_\gamma f(z)dz=\int_\gamma[\overline{z}+g(z)]dz=\oint_\gamma \overline{z}\,dz+\oint_\gamma g(z)dz=\oint_\gamma \overline{z}\,dz+0=$$

$$=\int_\gamma x\,dx+y\,dy+i\int_\gamma x\,dy-y\,dx=U+iV.$$

If (γ) smooth the function $U=\int_\gamma x\,dx+y\,dy=0$ according to the Stokes theorem and $K(p)=2i$ so $E(p)=2i\,dA$.

We could interpret this result in the following way: The $f\ (z)$ function generates two geometries: one of constante curvature - null - and an imaginary one having complex constant curvature purely imaginary. An other situation would be if $K(p)=2i\frac{\partial f}{\partial \overline{z}}$ a complex function $K(p)=U(p)+iV(p)$ of point p. In this situation $U(p)$ and $V(p)$ correspond to some geometries on surfaces with variable (non constant) curvature, obtained by a bijection of stereographic projections type.

How do we characterize the holomorphe functions in the vecinity of isolated singularities?

We'll take an example:

$$f(z)=\frac{1}{z},\ z=0 \text{ simple pole}$$

In this situation $f(z)=\frac{1}{z}=\frac{\overline{z}}{|z|^2}$ and $K(p)=\frac{2i}{|z|^2}$. If we surround the pole with a disk of ε ray centered in it, then $K(p)=\frac{2i}{\varepsilon^2}$ and $E(p)=2\pi i$.

The case of essential singularities is more complicated because in their vecinity the function can take any value (with one exception).

It seems we can construct double geometries for any complex holomorphe or non function on surfaces with constant or non-constant curvature.

I find that the most interesting fact is the possibility of studying the movement on a surface with non-constant curvature (Σ) with the help of some complex functions with areolar differential (complex curvature) a function $K(p)=U(p)+iV(p)$, p - a point in the plane $\mathbb{C}$ (complex plane) obtained by a bijection convenient to the surface (Σ).

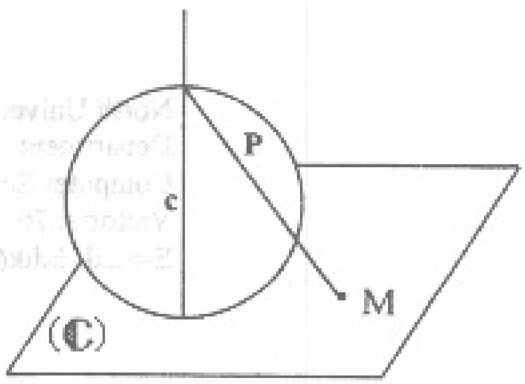


Figure 5

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