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SOLUTIONS FOR AN EQUATION OF A BEND OF A WEIGHABLE ROD

Gennady ULITIN

Abstract. Two approaches to a solution of a linear differential equation escribing Bend of a weighable rod are investigated. The solutions are represented in special functions. Practical application of them is shown.
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An equation of a bend of an elastic weighable rod with a length l being under the action of longitudinal force N can be reduced to

$$\frac{d^2u}{d\xi^2} \pm a^2\xi u = Ra^2, \qquad (1)$$

where $\xi = N + ql, q$ – weight of a length unit, x – an axial coordinate, a – a constant value, $u = y'_x$, $y(x)$ – a shape of bend for a rod, R – a reaction of an upper fulcrum. The sign plus corresponds to a condition of compression, the sign minus corresponds to straining.

The equation

$$\frac{d^2u}{d\xi^2} \pm a^2\xi u = 0 \qquad (2$$

is correspondent homogeneous to (1).

First we will investigate a case corresponding to compression. The solution of (2) can be represented in Bessel's functions [1]

$$u(\xi) = C_1\xi^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right) + C_2\xi^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right). \qquad (3)$$

A wronskian for a fundamental system of functions

$$\left\{\xi^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right);\xi^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)\right\}$$

taking in account $w\left(J_{\frac{1}{3}},J_{-\frac{1}{3}}\right)=-\frac{3\sqrt{3}\xi^{\frac{3}{2}}}{2\pi}$ see [2] will be $w=-\frac{3\sqrt{3}}{2\pi}$.

Using a variation of arbitrary constants method one can obtain a general solution for the equation (1) in the form

$$u(\xi)=C_1\xi^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)+C_2\xi^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)+$$

$$+\frac{2\pi R}{3\sqrt{3}}\left(\xi^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)\int_0^{\xi}\xi^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)d\xi-\xi^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)\int_0^{\xi}\xi^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2a}{3}\xi^{\frac{3}{2}}\right)d\xi\right). \quad (4)$$

The expression (4) can be simplified by introduction of the Lommel's functions $S_{\mu,\nu}(z)$ [2] and the new variable $z=\frac{2a}{3}\xi^{\frac{3}{2}}$. It gives us

$$u(z)=C_1z^{\frac{1}{3}}J_{\frac{1}{3}}(z)+C_2z^{\frac{1}{3}}J_{-\frac{1}{3}}(z)+R\left(\frac{2a}{3}\right)^{\frac{2}{3}}z^{\frac{1}{3}}S_{0,\frac{1}{3}}(z). \quad (5)$$

After a transition to a variable z in the substitution $y'_x=u$ and integration of (5) the shape of bend for the rod takes in the form

$$y(z)=C_1\bar{J}_{\frac{1}{3}}(z)+C_2\bar{J}_{-\frac{1}{3}}(z)+\frac{2R}{3q}\bar{S}_{0,\frac{1}{3}}(z)+C_3, \quad (6)$$

where $\bar{J}_{\pm\frac{1}{3}}(z)=\int_0^z J_{\pm\frac{1}{3}}(z)dz,\ \bar{S}_{0,\frac{1}{3}}(z)=\int_0^z S_{0,\frac{1}{3}}(z)dz.$

One can find the values $y'_x, y''_{xx}, y'''_{xxx}$ from (6) using recursion relations for Bessel's and Lommel's functions [2].

These formulas are needed to satisfy four boundary conditions which give a system of equations in unknowns C_1, C_2, C_3, R.

For a case of strain $\xi=qx-ql+N$ Bessel's and Lommel's functions are needed to be substituted for corresponding modified functions. In this case a solution of the equation (2) can be in Aire functions $Ai(z)$ and $Bi(z)$ [3]

$$u(z) = C_1 Ai(z) + C_2 Bi(z), \quad z = a^{\frac{2}{3}}\xi.$$

Here $w(Ai, Bi) = \pi^{-1}$. In an analogous manner using variation of arbitrary constants we will find the general solution of the equation (1)

$$u(z) = C_1 Ai(z) + C_2 Bi(z) - R\pi^{\frac{2}{3}} Gi(z), \qquad (7)$$

where $Gi(z)$ is special function [3].

Then it is possible to find an expression for bend $y(x)$ and its derivatives. For a case of compression the substitution $z = -z$ is needed. But this approach (7) to the solution is not so convenient as there are no recurrent formulas for derivatives of Airy functions.

To demonstrate an example we will solve a problem on stability of a weighable rod with fixed endpoints: $y(0) = y(l) = 0;\ y'(0) = y'(l) = 0$. Boundary conditions for formula (6) give $C_2 = C_3 = 0$. Using the situation that the determinant of the system for unknowns C_1 and R is equal to zero we obtain the equation of stability of the rod

$$\bar{J}_{\frac{1}{3}}(\alpha) S_{0,\frac{2}{3}}(\alpha) - J_{\frac{1}{3}}(\alpha) \bar{S}_{0,\frac{1}{3}}(\alpha) = 0. \qquad (8)$$

Minimal positive root α_1 of equation (8) defines the critical length

$$l_{kp} = \frac{1}{q}\left(\frac{3\alpha_1}{2a}\right)^{\frac{2}{3}}.$$

The solutions of the equation (1) in forms (5) and (7) enables us to investigate tensed and deformated state of a weighable rod in analytic expressions and to satisfy different boundary conditions. Besides, taking in account asymptotic representations for the special functions for large values of argument one can investigate stability of long rods. This is sufficiently difficult problem. So, for example, using asymptotic formulas for Bessel's and Lommel's functions we have

$$tg\left(\alpha - \frac{5\pi}{12}\right) = \alpha \ln \alpha,$$

which gives us $\alpha_1 = 2{,}454$.

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Donetsk National Technical University,
Department of Mathematics,
Artema str., 58,
Donetsk-00, 83000,
UKRAINE
e-mail: yuri@nosenko.dgtu.donetsk.ua