Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, XVII(2001), Nr. 1-2, 173–185

ON THE ASYMPTOTIC ANALYSIS OF THE FUNCTIONS OF DISCRETE VARIABLE

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Abstract. In this paper we give a sketch of an Asymptotic Analysis of the (real) functions of natural variable, i.e. of the real sequences. In the first part we mention a set of inequalities which conducts to some developments of the first order and in the second part we mention some (infinite) asymptotic developments. All these facts conducts to a general theory of the asymptotic developments.

2000 M.S.C.: 26D15, 30B10, 33F05, 40A05, 41A60.

Keywords: Sequence, series, limit of a sequence, sum of a series, order of convergence, the symbols O and o of Landau, asymptotic equivalence, asymptotic development.

1. Introduction. The mathematical facts which are conducted to the idea to elaborate the present sketch of the asymptotic analysis of the functions of discrete variable are of two kinds, which we present here.

In the first cathegory we have certain inequalities which characterize the „velocity" of convergence, of some sequences to their limit, as, for example:

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1}, \qquad ([33], [45], [54]),$$

or

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n} \qquad ([44])$$

(where $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$ and $\gamma = \lim_{n\to\infty} \gamma_n = 0,577\ldots$, is the well-known constant of Euler).

Such inequalities permits to obtain the first order of convergence of the respective sequences and also the first iterated limit: $\lim_{n\to\infty} n\left(e - (1+1/n)^n\right) = \frac{e}{2}$, respectively $\lim_{n\to\infty} n(\gamma_n - \gamma) = \frac{1}{2}$, conducting to a so called Asymptotic Analysis of the first order. The notion of order of convergence was rigurously precised by V.Berinde in [5]. Also, in this cathegory we have certain asymptotic developments as (2.20) – (2.28) and (2.29).

In the second cathegory we have the asymptotic developments, for example, that of Stirling:

$$\log n! = \log\sqrt{2\pi} + \left(n+\frac{1}{2}\right)\log n - n + \sum_{k=1}^{\infty}\frac{B_{k+1}}{k(k+1)n^k}$$

which is equivalent with

$$n! = \sqrt{2\pi n}n^n e^{-n}\exp\left(\sum_{k=1}^{\infty}\frac{B_{k+1}}{k(k+1)n^k}\right) \qquad ([23], [38])$$

or the asymptotic development of the harmonic sum $H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$,

$$H_n = \log n + \gamma + \frac{1}{2n} + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{B_k}{kn^k} \qquad ([23], [39]),$$

where B_k are the numbers of Bernoulli.

The aim of this paper is to put in evidence a certain number of inequalities, respectively developments of this nature, which may be able to construct a first collection of facts, serving to establish an Asymptotic Analysis of the function of discrete variable.

Generally speaking about the Asymptotic Analysis, considerated in all this generality, i.e. concerning the asymptotic study of the functions having the variable in the real or complex continuum, we remember that it has begun since to L.Euler, but its modern form was given by Th.Stieltjes (1886) and H.Poincaré (also 1886); the most important modern texts in this matter, are the works of G.H.Hardy [21], J.G.Van der Corput ([9] – [14]), N.G.De Bruijn [6], E.Copson [8], A.Erdely [17]. We also find useful informations in D.E.Knuth [23], R.L.Graham, D.E.Knuth and O.Patashnik [20], F.Olver [29], J.-L. Ovaert and J.-L.Verley [30], A.Georgescu [19]. We also mention two treatises of Mathematical Analysis: E.Whittaker & G.N.Watson [58] and J.Dieudonné [16] in which the problem of the Asymptotic Analysis is examineted more detailed.

The considerations of this paper are concerning a particular case of the Asymptotic Analysis, when the domain of the variable is N, whith the unique accumulation point ∞.

We remember, accoring to V.Berinde [5] that, if $(u_n)_n$ and $(v_n)_n$ are two convergent sequences, having the limits u, respective v, then, supposing that the limit exists and $\lim\limits_{n\to\infty}\frac{|u_n-u|}{|v_n-v|}=l$, we have:

a) The sequence $(u_n)_n$ converges faster (to u) than the sequence $(v_n)_n$ (to v), if $l=0$.

b) The two sequences have the same order of convergences if $l\neq 0$ and $l\neq\infty$.

We write that $u_n = O(v_n)$ if it exists a number $A>0$, so that $|u_n|\le A|v_n|$, for all $n\in N$; analogously, we also write that $u_n = o(v_n)$ if $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=0$. In particular, we denote by $o(1)$ an expression which tende to 0 for $n\to\infty$. The sequences $(u_n)_n$ and $(v_n)_n$ are called asympotically equivalent if $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=1$; we write $u_n\sim v_n$.

Consider a real function of natural variable (sequence) $(a_n)_n$ and let $(u_k)_k$ be a sequence of functions of natural variable n, so that $u_{k+1} = o(u_k)$ for all $k = 0, 1, 2, \ldots, n$. A series of functions $\sum_{k=0}^{\infty} \alpha_k u_k(n)$, with $u_{k+1} = o(u_k)$, is called an asymptotic series. If it exists a sequence of constants $(c_k)_k$, so that $a_n \sim c_0 u_0(n) + c_1 u_1(n) + \ldots + c_k u_k(n)$ for all $k = 0, 1, 2, \ldots, n$, we call according to H.Poincaré, the series $\sum_{k=0}^{\infty} c_k u_k(n)$ an asymptotic development of the function f. The sequence of functions of natural variable n, $(u_k)_k$, $n \mapsto u_k(n)$ is called the sequence of functions after which we make the development, and the coefficients $c_0, c_1, c_2, \ldots$ are called the coefficients of the development, or, still, the iterated limits of the function of natural variable (of the sequence) $(a_n)_n$, because they are given by the successive formula:

$$c_0 = \lim_{n\to\infty} \frac{a_n}{u_0(n)};\ c_1 = \lim_{n\to\infty} \frac{a_n - c_0 u_0(n)}{u_1(n)};\ c_2 = \lim_{n\to\infty} \frac{a_n - c_0 u_0(n) - c_1 u_1(n)}{u_2(n)} \text{ etc.}$$

An asymptotic development attached to a function of natural variable $(a_n)_n$ can be divergent or convergent for $k \to \infty$, and, in the case of convergence, can converge for $k \to \infty$ to the function of natural variable $(a_n)_n$ or to another function of natural variable.

2. Some characterizations of the order of convergence of certain sequences by inequalities

Let $(a_n)_n$ be a convergent sequence of real numbers, $a = \lim_{n\to\infty} a_n$ and $(u_n)_n$, $(v_n)_n$ two sequences of real numbers, different of zero, but converging to zero, with $u_n < v_n$ for all $n \in N$. If $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ (i.e.$u_n \sim v_n$), we will say that the double inequality

$$u_n < a - a_n < v_n \quad (\text{if } a_n < a)$$

respectively

$$u_n < a - a_n < v_n \quad (\text{if } a < a_n)^{1)}$$

characterizes the order of convergence of the sequence $(a_n)_n$ to its limit a.

This terminology is justified by the fact that dividing the first double inequality by u_n, we obtain:

$$1 < \frac{a_n - a}{u_n} < \frac{v_n}{u_n},$$

from which, it follows $\lim_{n\to\infty} \frac{a - a_n}{u_n} = 1^{2)}$. (A similar remark is valuable

[1] The inequality $a_n < a$ is also satisfied in the particular case in which the sequence $(a_n)_n$ is strictly increasing; analogously, the inequality $a < a_n$ is satisfied if the sequence $(a_n)_n$ is strictly decreasing.

[2] We obtain a similar result dividing by v_n.

Let $(\Omega_n)_n$ be the sequence of general term $\Omega_n = \dfrac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}$ (see [27], pag. 218). We have the inequalities:

(2.7) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi n}}$$ (J.Wallis) ([27])

(2.8) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}$$ (1956, [22])

(2.9) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}$$ ([32])

All conducts to the result $\lim_{n\to\infty} \Omega_n\sqrt{n} = \frac{1}{\sqrt{\pi}}$, but (2.9) already prefigures the beginning of an asymptotic development. We also have the inequalities:

(2.10) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32(n+1/4)}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32(n+\theta)}\right)}}; \ (\theta > \frac{1}{4})$$ ([50])

(2.11) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < \Omega_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}\right)}}$$ ([50])

which confirm the fact that in (2.9) the term $\frac{1}{32n}$ is correct and also show the next term of the prefigurated asymptotic series.

Let $W_n = \dfrac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot \ldots \cdot (2n-1)(2n+1)} = \dfrac{1}{\Omega_n^2}\cdot\dfrac{1}{2n+1}$ be the sequence of Wallis, which converges to $\pi/2$.

We have the inequality:

(2.12) $$\frac{\pi}{4(2n+1)}\left(1-\frac{1}{8n}\right) < \frac{\pi}{2} - W_n < \frac{\pi}{4(2n+1)} \ ([46]) \Rightarrow$$

$$\lim_{n\to\infty} n\left(\frac{\pi}{2}-W_n\right) = \frac{\pi}{8}$$

We have:

(2.13) $$\frac{1}{4n+2} < \log 2 - A_n < \frac{1}{4n+1} \quad ([48]) \Rightarrow \lim_{n\to\infty} n\,(\log 2 - A_n) = \frac{1}{4}$$

(where $A_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$).

Concerning the alternate harmonic series and the series of Leibniz, we have the inequalities:

(2.14) $$\frac{1}{2n+2} < (-1)^n\left(\log 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n}\right)\right) < \frac{1}{2n+1}$$
([52])

(2.15) $$\frac{1}{4n+1} < (-1)^n\left(\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{2n-1}\right)\right) < \frac{1}{4n}$$
([52])

These two inequalities give two obvious limits, one equal to $\frac{1}{2}$ and another equal to $\frac{1}{4}$.

Now let $L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$ $(n \geq 2)$ be; this sequence was considerated for the first time by the romanian mathematician Traian Lalescu, in 1901. We have the inequality:

(2.16) $$\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} < L_n < \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$ ([26])

Let now $R_n = H_n - \log(n + 1/2)$; therefore:

(2.17) $$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2} \quad ([15]) \quad \Rightarrow \lim_{n\to\infty} n^2(R_n - \gamma) = \frac{1}{24}$$

If $T_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24n}\right)$, therefore:

(2.18) $$\frac{1}{48(n+1)^3} < \gamma - T_n \leq \frac{1}{48n^3} \quad ([28]) \quad \Rightarrow \lim_{n\to\infty} n^3(\gamma - T_n) = \frac{1}{48}$$

Now, in the classical sequence of the Euler's constant, we don't modify the logarithmic term, but the last term of the harmonic sum, passing $\frac{1}{n}$ in $\frac{1}{2n}$. We denote $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \log n = \left(\gamma_n - \frac{1}{2n}\right)$. So, we obtain a convergence to γ, which also has the order $\frac{1}{n^2}$, namely

(2.19) $$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2} \quad ([51]) \quad \Rightarrow \lim_{n\to\infty} n^2(\gamma - x_n) = \frac{1}{12}.$$

Now, let be $r \in N^*$, $\alpha = 1, 2, 3, \ldots, r$. Let note

$$H_{n,\alpha}^{(r)} = \frac{1}{\alpha} + \frac{1}{\alpha + r} + \frac{1}{\alpha + 2r} + \ldots + \frac{1}{\alpha + (n-1)r}.$$

In [53], using the theory of the complex functions, we have established the following equalities:

a) For the case $r = 2$:

(2.20) $$H_{n,1}^{(2)} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} = \frac{1}{2}\log n + \frac{1}{2}(\gamma + \log 2) + o(1).$$

(2.21) $$H_{n,2}^{(2)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} = \frac{1}{2}\log 2 + \frac{1}{2}\gamma + o(1).$$

b) For the case $r = 3$:

(2.22) $$H_{n,1}^{(3)} = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n-2} = \frac{1}{3}\log n + \frac{1}{3}\left(\gamma + \frac{3}{2}\log 3 + \frac{\pi\sqrt{3}}{6}\right) + o(1).$$

(2.23) $$H_{n,2}^{(3)} = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3n-1} = \frac{1}{3}\log n + \frac{1}{3}\left(\gamma + \frac{3}{2}\log 3 - \frac{\pi\sqrt{3}}{6}\right) + o(1).$$

(2.24) $$H_{n,3}^{(3)} = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \ldots + \frac{1}{3n} = \frac{1}{3}\log n + \frac{1}{3}\log n + \frac{1}{3}\gamma + o(1).$$

c) For the case $r = 4$:

(2.25) $$H_{n,1}^{(4)} = 1 + \frac{1}{5} + \frac{1}{9} + \ldots + \frac{1}{4n-3} = \frac{1}{4}\log n + \frac{1}{4}\left(\gamma + 3\log 2 + \frac{\pi}{2}\right) + o(1).$$

(2.26) $$H_{n,2}^{(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{1}{4n-2} = \frac{1}{4}\log n + \frac{1}{4}(\gamma + 2\log 2) + o(1).$$

(2.27) $$H_{n,3}^{(4)} = \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{4n-1} = \frac{1}{4}\log n + \frac{1}{4}\left(\gamma + 3\log 2 - \frac{\pi}{2}\right) + o(1).$$

(2.28) $$H_{n,4}^{(4)} = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \ldots + \frac{1}{4n} = \frac{1}{4}\log n + \frac{1}{4}\gamma + o(1),$$

which give the principal term of the mentioned (divergent) sums and also the limit of the sequences of general term $x_{n,\alpha}^{(r)} = H_{n,\alpha}^{(r)} - \frac{1}{r}\log n$.

Consider now $S_n = \log_2 3 + \log_3 4 + \ldots \log_n(n+1)$, $(n \geq 2)$ (a sum introdeced for the first time by L.Panaitopol in the work [31]). In [56] we have established the asymptotic formula of the first order

(2.29) $$S_n = (n-1) + \log(\log n) + A + o(1),$$

where $A = \alpha + \beta$, $\alpha = \lim_{n\to\infty}\left(S_n - (n-1) - \sum_{k=2}^{n} 1/k\log k\right) = -0,24284\ldots$,

$\beta = \lim_{n\to\infty}\left(\sum_{k=2}^{n} 1/k\log k - \log(\log n)\right) = 0,79468\ldots$, therefore $A = 0,55184\ldots$

3. Some asymptotic developments of certain classical sequences

A deeper revelation of the analytical structure of certain functions of natural variable (sequences of real numbers) are given by some asymptotic developments (according to certain asymptotic sequences).

The main posibilities to determine these asymptotic sequences and also of the coefficients are given by:

– the use of the taylorian developments and of the operations with formal series;

– the use of the multinomial formula;

– the succesive calculation of the coefficients using Stolz-Cesàro lemma in the case $\frac{0}{0}$, passing, if is necessary, to the real variable;

– the use of the summation formula (e.g. the summation formula of Euler-Mac Laurin).

In the Introduction we have presented two asymptotic developments. We continue here presenting other some one developments. The first of these, obtained in [2] for the real positive variable x, conducts (remplacing x by $\frac{1}{n}$, where $n \in \mathbb{N}^*$) to:

$$\left(1+\frac{1}{n}\right)^n = e\left(a_0 + a_1\frac{1}{n} + a_2\frac{1}{n^2} + \dots\right), \tag{3.1}$$

where the rational constants $a_0, a_1, a_2, \dots, a_n$ are given by the formula

$$a_0 = 1 \quad \text{and} \quad a_i = (-1)^k \sum_{(C)} \frac{1}{k_1!k_2!\dots k_i! 2^{k_1} 3^{k_2} \dots (i+1)^{k_i}}$$

(where the condition (C) is $k_1 + 2k_2 + \dots + ik_i = 1$),for all $i \geq 1$, the sum concerning all the integer nonegative solutions $(k_1, k_2, \dots, k_i)$ of the equation $k_1 + 2k_2 + \dots + ik_i = 1$.

In particular, computing the first constants k_i, the autors obtain $a_1 = -\frac{1}{2}$, $a_2 = \frac{11}{24}$, $a_3 = -\frac{7}{16}$, $a_4 = \frac{2447}{5760}$, $a_5 = -\frac{959}{2304n^5}$ etc. Therefore

$$\left(1+\frac{1}{n}\right)^n = e\left(1 - \frac{1}{2n} + \frac{11}{24n^2} - \frac{7}{16n^3} + \frac{2447}{5766n^4} - \frac{959}{2304n^5} + \dots\right) \tag{3.1'}$$

Another formula was obtained in [40] and concernes the asymptotic development of the sequence of Wallis

$$W_n = \frac{\pi}{2}\exp\left(\frac{a_1}{n} + \frac{a_2}{n^2} + \dots + \frac{a_k}{n^k} + \dots\right), \tag{3.2}$$

where the rational coefficients $a_1, a_2, \dots, a_k, \dots$ are given by the formula:

$$a_k = \begin{cases} \dfrac{1}{k \cdot 2^k}, & \text{for } k = 0, \\ \dfrac{2(2^{k+1}-1)B_{k+1} - (k+1)}{k(k+1)\,2^k}, & \text{for } k \geq 1, \end{cases}$$

where B_k are the numbers of Bernoulli. In the same work we have also obtained the second asymptotic development

$$W_n = \frac{\pi}{2}\left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \ldots + \frac{b_k}{n^k} + \ldots\right), \tag{3.3}$$

where the rational coefficients $b_1, b_2, \ldots, b_k, \ldots$ are defined by the formula:

$$b_k = \sum_{(C)} \frac{a_1^{i_1} a_2^{i_2} \cdot \ldots \cdot a_k^{i_k}}{i_1! i_2! \cdot \ldots \cdot i_k!}$$

(where the condition (C) signifies $i_1 + 2i_2 + 3i_3 + \ldots + ki_k = k$), the sum concerning all the positive solutions of the mentioned equation and $a_1, a_2, \ldots, a_k, \ldots$ being the constants from (3.2). So we have obtained:

$$W_n = \frac{\pi}{2}\left(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \ldots\right). \tag{3.3'}$$ [1]

In the same paper, we also have deduced from (3.31) the asymptotic development:

$$\Omega_n = \frac{1}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} + \ldots\right). \tag{3.4}$$

A development of a more special form containing a square root, which improve al the inequalities (2.7) – (2.11) and in fact explains from a deeper analytic point of view their succesion is:

$$\Omega_n = \frac{1}{\sqrt{\pi\left(n - \dfrac{1}{4} + \dfrac{1}{32n} - \dfrac{1}{128n^2} - \dfrac{5}{2048n^3} + \dfrac{23}{4096n^4} + \ldots\right)}} \tag{3.5}$$

(from [51]). In this work also I have obtained the corresponding asymptotic development which implies a root of order four; this explains certain inequalities of J.Gurland, R.E.Shafer, J.Grimland jr. and S.Glidewell, mentioned in the work. This development is:

$$\Omega_n = \frac{1}{\sqrt{\pi}\sqrt[4]{n^2 + \dfrac{n}{2} + \dfrac{1}{8} - \dfrac{1}{32n} - \dfrac{1}{256n^2} + \ldots}} \tag{3.6}$$

In order to deep the development of H_n, sugested by the sequence of (1.17) and (1.18), I have obtained in [56] the following asymptotic development:

[1] By a typographical mistake the coefficient b_2 was printed in [41] as $\frac{3}{32}$; its correct value is $\frac{5}{32}$.

$$H_n = \gamma + \log\left(n - \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \dots\right) \tag{3.7}$$

We do mention that there were also established the asymptotic development for the expressions $x_n = 1^1 2^2 3^3 \dots n^n$ and $y_n = \dfrac{1 \cdot 2^2 \cdot 3^3 \cdot \dots \cdot n^n}{1^n \cdot 2^{n-1} \cdot 3^{n-2} \cdot \dots \cdot (n-1)^2 \cdot n}$ (A. Lupaş and L.Lupaş [25]).

We conclude mentioning that there were researched asymptotic developments for the sequence $(L_n)_{n\geq 1}$ of general term $L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$, introducted by Traian Lalescu in 1901. This study was initiated by the regreted Şerban-Nicolae Buzeţeanu in his thesis

[7]. We do present herein the asymptotic development obtained by L. Tóth and M.Bencze in [22]. It has a very special form, the sequence after which the development was made being for $k \geq 2$ an interesting combination betwen the logarithm and the negative powers of n, namely:

$$\frac{1}{n};$$

$$\frac{\log^2 n}{n^2};\quad \frac{\log n}{n^2};\quad \frac{1}{n^2};$$

$$\frac{\log^3 n}{n^3};\quad \frac{\log^2 n}{n^3};\quad \frac{\log n}{n^3};\quad \frac{1}{n^3};\ \text{etc}$$

This development is:

$$L_n = \frac{1}{e}\left(1 + \frac{1}{2n} - \frac{\log^2 n}{8n^2} - \frac{(2A-1)\log n}{4n^2} - \frac{3A^2 - 3A + 2}{6n^2} - \frac{\log^3 n}{24n^3} - \right.$$
$$\left. - \frac{(4A-3)\log^2 n}{16n^3} - \frac{(12A^2 - 18A + 11)\log n}{24n^3} - \frac{12A^3 - 27A^2 + 33A - 15}{360n^3} + \dots\right), \tag{3.8}$$

where $A = \log\sqrt{2\pi} = 0,39908\dots$ is the so called Stirling constant (it is also mentioned in [32]).

All these considerations lead to the idea of reinforcement of the existing material, whithing an Asymptotic Analysis of the functions of the natural variable. All these will be presented more detailed in a future work.

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Received: 3.09.2001

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