Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVIII(2002), Nr. 1, 1-6

Dedicated to Costică MUSTĂŢA on his 60^{th} anniversary

ON THE SEQUENCE OF GBS OPERATORS OF STANCU-TYPE

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Abstract. We discuss about the approximation order of a B-continuous functions by the GBS operators of Stancu

MSC: 41A17, 41A25, 41A36
Keywords: linear positive operator, Bernstein's operator, Stancu's operator, boolean sum operator, parametric extension, Korovkin-type theorem, B-continuous function, mixed modulus of smoothness

1. Preliminaries

In 1969 D.D. Stancu ([6]) introduced and studied the positive and linear operator $P_m^{(\alpha,\beta)}$, depending on two real non-negative parameters α and β which satisfy the condition $0 \leq \alpha \leq \beta$. This operator is defined on $C([0,1])$ and associates to any function $f \in C([0,1])$ the polynomial

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum\nolimits_{k=0}^{m} p_{m,k}(x) \cdot f\left(\frac{k+\alpha}{m+\beta}\right). \tag{1}$$

Note that in (1.1) $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, i.e.

$$p_{m,k}(x) = \left(\begin{array}{c} m \\ k \end{array}\right) x^k (1-x)^{m-k} \tag{2}$$

D.D. Stancu proved in ([6]) a convergence theorem for the sequence $\left(P_m^{(\alpha,\beta)} f\right)_{m\in\mathbb{N}}$ and discussed about the approximation order of a function $f \in C([0,1])$ by $P_m^{(\alpha,\beta)} f$, using the first order modulus of smoothness.

The notion of B-continuous function was introduced by K. Bögel ([4]).

Using the method of parametric extensions, we constructed in ([3]) a GBS operator of Stancu type, defined on the space $C_b(I^2)$ of B-continuous functions on the unit square $I^2 = [0,1] \times [0,1]$.

More exactly, in ([3]) we introduced the sequence of GBS operators of Stancu-type $(S_{m,n})_{m,n\in\mathbb{N}}$, where $S_{m\ ,n} : C_b(I^2) \to C_b(I^2)$ associates to any function $f \in C_b(I^2)$ the pseudopolynomials:

(3) $$(S_{m,n}\ f)(x,y) = \sum_{k=0}^{m}\sum_{l=0}^{n} p_{m,k}(x)p_{n,l}(y)\left\{f\left(\tfrac{k+\alpha_1}{m+\beta_1}, y\right) + \right.$$
$$\left. + f\left(x, \tfrac{l+\alpha_2}{n+\beta_2}\right) - f\left(\tfrac{k+\alpha_1}{m+\beta_1},\ \tfrac{l+\alpha_2}{n+\beta_2}\right)\right\}.$$

In (1.3) $\alpha_1, \beta_1, \alpha_2, \beta_2$ are real non-negative parameters satisfying the conditions $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$.

Note that for $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 0$ the operator $S_{m.n}$ reduces to the classical GBS operator of Bernstein.

Applying the Korovkin-type theorem for the approximation of B-continuous functions due to Badea C., Badea I., and Gonska H.H., (see([2]), in ([3]) we established the following result:

Theorem 1.1. *The sequence $(S_{m,n})_{m,n\in\mathbb{N}}$ converges to f, uniformly on I^2 as m and n tend to infinity, for any $f \in C_b(I^2)$.*

Using the mixed modulus of smoothness w_{mixed} (for this concept see ([2])), in the same paper ([3)], we proved the results contained in the following two theorems

Theorem 1.2. *For any $f \in C_b(I^2)$, any $\delta_1, \delta_2 > 0$ and any $(x,y) \in I^2$, the inequality:*

$$|(S_{m,n})(x,y) - f(x,y)| \le$$
$$\le \{1 + \frac{1}{2\delta_1(m+\beta_1)}\sqrt{m + 4(\alpha_1 - \beta_1 x)^2} + \frac{1}{2\delta_2(n+\beta_2)}\sqrt{n + 4(\alpha_2 - \beta_2 y)^2} +$$

(4) $$+\frac{1}{4\delta_1\delta_2(m+\beta_1)(n+\beta_2)}\sqrt{(m + 4(\alpha_1 - \beta_1 x)^2)(n + 4(\alpha_2 - \beta_2 y)^2)}\}w_{mixed}(\delta_1, \delta_2)$$

holds.

Theorem 1.3. *For any $f \in C_b(I^2)$, and any $(x,y) \in I^2$, the inequality:*

(5) $$|(S_{m,n}\ f)(x,y) - f(x,y)| \le \frac{9}{4} w_{mixed}\left(\frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1}, \frac{\sqrt{n+4\alpha_2^2}}{m+\beta_2}\right)$$

holds.

The aim of the present paper is to give some refinements of the inequalities (1.4) and (1.5)

2. Main results

Theorem 2.1. *If:*
(i) $\alpha_1 \in [0, \frac{1}{4}]$, $\beta_1 \in [\alpha_1, 2\alpha_1]$, $\alpha_2 \in [0, \frac{1}{4}]$, $\beta_2 \in [\alpha_2, 2\alpha_2]$,
or
(ii) $\alpha_1 \in [\frac{1}{4}, \frac{1}{2}]$, $\beta_1 \in [4\alpha_1^2, 2\alpha_1]$, $\alpha_2 \in [\frac{1}{4}, \frac{1}{2}]$, $\beta_2 \in [4\alpha_2^2, 2\alpha_2]$,
the inequality

(6) $$|(S_{m,n}\, f)(x,y) - f(x,y)| \le \frac{9}{4} w_{mixed}\left(\frac{1}{\sqrt{m+4\alpha_1^2}}, \frac{1}{\sqrt{n+4\alpha_2^2}}\right)$$

holds, for any $f \in C_b(I^2)$ *and any* $(x,y) \in I^2$.

Proof. From the properties of the mixed modulus of smoothness and from the Theorem 1.3, it is suficient to prove that under the hypothesis (i) or (ii) we have

(7) $$\frac{\sqrt{m+4\alpha_1^2}}{m+\beta_1} \le \frac{1}{\sqrt{m+4\alpha_1^2}}, \frac{\sqrt{n+4\alpha_2^2}}{m+\beta_2} \le \frac{1}{\sqrt{n+4\alpha_2^2}}.$$

The first inequality in (2.2) can be written in the form

(8) $$4\alpha_1^2 \le \beta_1.$$

Let us to suppose that $0 \le \alpha_1 \le \beta_1 \le 2\alpha_1$. Then (2.3) is possible if and only if $4\alpha_1^2 \le 2\alpha_1$, i.e. $\alpha_1 \in \left[0, \frac{1}{2}\right]$. For the quantity $4\alpha_1^2$, we have the possibilities $4\alpha_1^2 \le \alpha_1$ or $\alpha_1 \le 4\alpha_1^2$. We get $\alpha_1 \in \left[0, \frac{1}{4}\right]$, respectively $\alpha_1 \in \left[\frac{1}{4}, \frac{1}{2}\right]$. So, we can conclude that for $\alpha_1 \in \left[0, \frac{1}{4}\right]$, $\beta_1 \in [\alpha_1, 2\alpha_1]$ or for $\alpha_1 \in \left[\frac{1}{4}, \frac{1}{2}\right]$, $\beta_1 \in [4\alpha_1^2, 2\alpha_1]$, the inequlity (2.3) holds.

The second inequality in (2.2) can be written in the form

(9) $$4\alpha_1^2 \le \beta_2.$$

Similar to the proof of (2.3), we get that for $\alpha_2 \in \left[0, \frac{1}{4}\right]$, $\beta_1 \in [\alpha_2, 2\alpha_2]$ or $\alpha_2 \in q$ $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\beta_2 \in [4\alpha_2^2, 2\alpha_2]$, the inequlity (2.4) holds.

In a similar way, starting with the Theorem 1.2, one proves

Theorem 2.2. *If*
(i) $\alpha_1 \in \left[\frac{1}{4}, \frac{1}{2}\right]$, $\beta_1 \in [\alpha_1, 4\alpha_1^2]$, $\alpha_2 \in \left[\frac{1}{4}, \frac{1}{2}\right]$, $\beta_2 \in [\alpha_2, 4\alpha_2^2]$,
or
(ii) $\alpha_1 \ge \frac{1}{2}$, $\beta_1 \in [\alpha_1, 2\alpha_1]$, $\alpha_2 \ge \frac{1}{2}$, $\beta_1 \in [\alpha_2, 2\alpha_2]$,
the inequality

(10) $$|(S_{m,n}f)(x,y) - f(x,y)| \le \left(1 + \frac{4\alpha_1^2+1}{2(\beta_1+1)}\right)\left(1 + \frac{4\alpha_2^2+1}{2(\beta_2+1)}\right) \times \times \omega_{mixed}\left(\frac{1}{\sqrt{m+4\alpha_1^2}}, \frac{1}{\sqrt{n+4\alpha_2^2}}\right)$$

holds, for any $f \in C_b(I^2)$ *and any* $(x, y) \in I^2$.

Theorem 2.3. *If*
(i) $\beta_1 \in \left[\frac{1}{4}, 1\right]$, $\alpha_1 \in \left[0, \beta_1 - \frac{\sqrt{\beta_1}}{2}\right]$, $\beta_2 \in \left[\frac{1}{4}, 1\right]$, $\alpha_2 \in \left[0, \beta_2 - \frac{\sqrt{\beta_2}}{2}\right]$
or
(ii) $\beta_1 \geq 1$, $\alpha_1 \in \left[0, \frac{\sqrt{\beta_1}}{2}\right]$, $\beta_2 \geq 1$, $\alpha_2 \in \left[0, \frac{\sqrt{\beta_2}}{2}\right]$
the inequality

$$|(S_{m,n}f\,)(x, y) - f(x, y)| \leq$$

$$\leq \left(1 + \frac{4(\beta_1 - \alpha_1)^2 + 1}{2(\beta_1 + 1)}\right)\left(1 + \frac{4(\beta_2 - \alpha_2)^2 + 1}{2(\beta_2 + 1)}\right) \times$$

$$\times \omega_{mixed}\left(\frac{1}{\sqrt{m + 4(\beta_1 - \alpha_1)^2}}, \frac{1}{\sqrt{n + 4(\beta_2 - \alpha_2)^2}}\right) \tag{11}$$

holds, for any $f \in C_b(I^2)$ *and any* $(x, y) \in I^2$.

Starting with the Theorem 1.3, one proves

Theorem 2.4. *If*

$$\beta_1 \leq 1\,,\ \ \alpha_1 \in \left[\beta_1 - \frac{\sqrt{\beta_1}}{2}, \frac{\beta_1}{2}\right] \cap [0, +\infty]$$

and

$$\beta_2 \leq 1\,,\ \ \alpha_2 \in \left[\beta_2 - \frac{\sqrt{\beta_2}}{2}, \frac{\beta_2}{2}\right] \cap [0, +\infty]$$

the inequality

$$|(S_{m,n}f\,)(x, y) - f(x, y)| \leq$$

$$\leq \frac{9}{4}\,\omega_{mixed}\left(\frac{1}{\sqrt{m + 4(\beta_1 - \alpha_1)^2}}, \frac{1}{\sqrt{n + 4(\beta_2 - \alpha_2)^2}}\right) \tag{12}$$

holds, for any $f \in C_b(I^2)$ *and any* $(x, y) \in I^2$.

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Received: 11.05.2002

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