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Dedicated to Costică MUSTĂŢA on his 60th anniversary

SUBCLASSES OF INTERSECTIONAL CONVEXITIES FOR SETS

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Abstract. In the general classification of the convexity properties for sets [4], the convexities defined as intersection of certain standard sets appear in more classes. First of all, we shall prove that all these convexity concepts are defined by segmential methods. The type of segmential method involved in the construction of an intersectional convexity lead to three classes of intersectional convexities.

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1. Intersectional approach

It is well known that each convex set in an n-dimensional linear space is the intersection of all half-spaces containing it. This is the starting point of the intersectional method of defining the property of convexity, which is an outer approach. The convexity properties defined by intersectional methods will be called in what follows intersectional convexities. The classification of 100 convexity properties contained in chapter 8 of our book G. Cristescu and L. Lupşa (2002) splits the intersectional convexities into three classes. But the general classification from chapter 8 of the above mentioned book divides a set of 100 convexity properties in classes, without looking to a possible subdivision according to other criteria. The first chapter of that book contains another classification of the convexity properties according to the manner in which it is possible to modify the definition of the classical convexity in order to obtain a new convexity that does not imply the connectivity. Three types of methods are identified: segmential methods (based on a notion of straight-line segment), intersectional methods (based on a notion of hull) and separational methods (based on separation properties). The main idea of this paper is to go further with the classification, identifying subclasses of the main classes of convexities from the general classification. They are described by considering the idea of the first chapter of the above-mentioned book. This paper describes the subclasses of intersectional convexities:

1. (S, s) intersectional convexity properties,
2. partial (a, e) - ((S, s), R) intersectional convexity properties,
3. (e, a) – ((S, S), r) intersectional convexity properties.

The (S, s) intersectional convexity properties are included in the class of (S, s) convexities. The partial (a, c) - ((S, s), R) intersectional convexity properties are elements of a subclass of the partial (a, c) - ((S, s), R) convexities. Also, the (c, a) – ((S, S), r) intersectional convexity is a subclass of (c, a) – ((S, S), r) convexity properties. We shall keep the notations of the general classification.

2. The class of (S, s) intersectional convexities

Let us consider a nonempty set X, a nonempty subset $S \subseteq 2^X$, and a function $s: S \to 2^X$. The elements of the subset S of the set of the parts of X will replace the pairs of points which appear in the definition of the classical convexity. Function $s: S \to 2^X$ will replace the straight-line segment determined by a pair of elements, which appear in the definition of the classical convexity.

Definition 2.1. *A set* $A \subseteq X$ *is said to be* (S, s) *convex if condition* $s(C) \subseteq A$ *is satisfied for each* $C \subseteq A$, $C \in S$.

Theorem 2.1. *If* $A \subseteq X$ *and* $B \subseteq X$ *are* (S, s) *convex sets then* $A \cap B$ *is* (S, s) *convex.*

Proof. If A is (S, s) convex then $s(C) \subseteq A$ is satisfied for each $C \subseteq A$, $C \in S$. If B is (S, s) convex then $s(C) \subseteq B$ is satisfied for each $C \subseteq B$, $C \in S$. Supposing that $C \in S$ and $C \subseteq A \cap B$, then $C \subseteq A$ implies $s(C) \subseteq A$ and $C \subseteq B$ implies $s(C) \subseteq B$. Then $s(C) \subseteq A \cap B$, meaning the (S, s) convexity of $A \cap B$.

In what follows we shall present all (S, s) intersectional convexities, together with the manner in which the elements S and s are constructed. In this case, function s is obtained as an intersection of all sets having certain properties and containing its argument. We call this subclass the strong intersectional convexities.

2.1. The convex hull approach of the convexity (g-convexity) (V. Soltan (1984))

First, we recall that in a nonempty set X, a mapping $g: 2^X \to 2^X$ is called the convex hull operator on X if:

i. $A \subseteq g(A)$, $g(g(A)) = g(A)$, $g(B) \subseteq g(A)$, for any $A, B \in 2^X$, $B \subseteq A$.

If g is a convex hull operator defined on X, and $G = \{A \subseteq X \mid g(A) = A\}$, then the pair (X, G) is a convexity space on X, called the convexity space generated by g. Here, the notion of convexity space is defined in the sense of D.C. Kay, E.W. Womble (1971).

Conversely, if (X, G) is a convexity space on X, then a convex hull operator on X is defined by the mapping $g: 2^X \to 2^X$ given by $g(A) = \cap \{C \in G \mid A \subseteq C\}$, for all $A \in 2^X$. This function g is called the convex hull operator generated by G.

If g is a convex hull operator defined on X then the straight-line segment determined by two points $x \in X$ and $y \in X$ is the set $<x, y> = g(\{x, y\})$.

As usual, a set $A \subseteq X$ is said to be *g-convex* if for every two points $x \in A$ and $y \in A$, the subset $<x, y>$ is included in A.

As above, we put X=Y, $S = \{\{x, y\} \mid x \in X, y \in X\}$ and we consider function $s: S \to 2^X$ defined by

$$s(\{x, y\}) = <x, y> = g(\{x, y\}) \text{ for } \{x, y\} \in S.$$

With these elements we can prove as above that the property of g - convexity is a (S, s) convexity. It is an intersectional convexity because

$$s(\{x, y\}) = g(\{x, y\}) = \cap \{C \in G \mid \{x, y\} \subseteq C\}.$$

2.2. ε – convexity (J. Perkal (1956))

Let (X,d) be a metric space. The ε – convex hull of $A \subseteq X$ is the set of all points $p \in X$ situated at a distance of at least ε/2 from each point situated at a distance greater than ε/2 from A. Simbol $C_\varepsilon(A)$ denotes the ε - convex hull of A. Obviously, if $A \subseteq B$ then $C_\varepsilon(A) \subseteq C_\varepsilon(B)$. On the other hand, for each $A \subseteq X$ the inclusion $A \subseteq C_\varepsilon(A)$ occurs.

A set $A \subseteq X$ is said to be ε – *convex* if $A = C_\varepsilon(A)$. Obviously, it is enough to say that a set $A \subseteq X$ is ε – convex if $C_\varepsilon(A) \subseteq A$.

If we take $X = Y$, $S = 2^X$ and $s: S \to 2^X$ is defined by $s(A) = C_\varepsilon(A)$ for each $A \subseteq X$, using the monotony of function s, it is easy to prove that a set is ε – convex if and only if it is (S, s) convex.

2.3. Polynomial convex sets (G. Stolzenberg (1962))

Let $E \subset C^n$ be a compact set. The polynomial hull of E is the set

$$\mathrm{hull}(E) = \{z \in C^n \mid |f(z)| = \max\{|f(p)| \mid p \in E\}, \forall \text{ polynomial } f \text{ on } C^n\}.$$

Set E is *polynomially convex* if E = hull(E).

Obviously, each set $E \subset C^n$ satisfies $E \subseteq \mathrm{hull}(E)$. Therefore, a set E is polynomially convex if and only if $\mathrm{hull}(E) \subseteq E$. On the other hand, it is evident that if $A \subseteq B$ then $\mathrm{hull}(A) \subseteq \mathrm{hull}(B)$.

Let us take $X = Y = C^n$, $S = 2^X$, $s: S \to 2^X$ defined by $s(A) = \mathrm{hull}(A)$ for each $A \subseteq C^n$, and $r = 1_{2^X}$. Therefore, a set $A \subset C^n$ is polynomially convex if and only if A is (S, s) convex.

2.4. Rational convex sets (G. Stolzenberg (1963))

Let $E \subset C^n$ be a compact set. The rational hull of E is the set

$$\text{R-hull}(E) = \{z \in C^n \mid |f(z)| = \max\{|f(p)| \mid p \in E\},$$

$$\forall \text{ rational function } f \text{ on } C^n, \text{ which is analytic about } E\}.$$

Set E is *rationally convex* if E = R-hull (E).

It is evident that if $A \subseteq B$ then R-hull (A) ⊆ R-hull (B). On the other hand, obviously, each set $E \subset C^n$ satisfies $E \subseteq$ R-hull (E). Therefore, a set E is rationally convex if and only if R-hull (E) ⊆ E.

Let us take $X = Y = C^n$, $S = 2^X$, function $s: S \to 2^X$ defined by $s(A) =$ R-hull (A) for each $A \subseteq C^n$, and $r = 1_{2^X}$. A set $A \subset C^n$ is rationally convex if and only if A is (S, s) convex.

3. The class of partial (a, e) - ((S, s), R) intersectional convexities

Let us consider a nonempty set X, a nonempty set Y, a nonempty subset $S \subseteq 2^X$, a function $s: S \to 2^Y$ and a family R of functions $r: 2^X \to 2^Y$. The significance of set S and function s are the same as in the previous section. Function $r: 2^X \to 2^Y$ will be used to rewrite the condition that every straight-line segment determined by pairs of elements of the set under consideration belong to this set. In the case of the classical convexity the R family is singleton and its unique element is the identity on 2^X.

Definition 3.1. *A set* $A \subseteq X$ *is said to be partially* (a, e) - ((S, s), R) *convex if there is a subset* $C \in S$, $C \subseteq A$, *such that there is a transformation* $r \in R$ *which satisfies the inclusion* $s(C) \sqsubseteq r(A)$.

Now we present all the intersectional convexity properties of this type. We call them partial weak intersectional convexities because the definition states that there is a family of subsets of certain type, which gives the required set by their intersection.

3.1. Convexity with respect to a family of sets (L. Danzer, B. Grunbaum, V. Klee (1963))

Let X be a nonempty set and a family of subsets $M \subset 2^X$. A set $A \subseteq X$ is said to be *convex with respect to* M (or, simply, M – *convex*) if there is a subfamily M' of M such that

$$A = \cap \{M \mid M \in M'\}.$$

Let us consider $Y = X \cup \{*\}$, where $* \notin X$, $S = 2^X$ and $s: S \to 2^Y$ defined by

$$s(B) = Y \text{ if there is not } M' \subset M \text{ such that } B = \cap \{M \mid M \in M'\},$$

$$s(B) = B \text{ if there is } M' \subset M \text{ such that } B = \cap \{M \mid M \in M'\}.$$

The family of transformations $R = \{r_W \mid w \in 2^X\}$ contains functions $r_W: 2^X \to 2^Y$ defined by $r_W(A) = \begin{cases} A, & W = A \\ \{*\}, & W \neq A \end{cases}$ for $A \in 2^X$.

Theorem 3.1. *A set* $A \subseteq X$ *is convex with respect to* M *if and only if* A *is partially* (a, e) - ((S, s), R) *convex.*

Proof. First, let us suppose that set A is convex with respect to M. We take C = A, therefore $C \in S$ and $C \subseteq A$. There is $M' \subset M$ such that $A = \cap \{M \mid M \in M'\}$. It means that $s(A) = A$. Taking W = A, we obtain $r_A(A) = A$ and $s(C) \subseteq A = r_A(A)$, implying that A is partially (a, e) - ((S, s), R) convex. Conversely, suppose that A is partially (a, e) - ((S, s), R) convex. Then there is $C \in S$ and $C \subseteq A$, such that $s(C) \sqsubseteq r(A)$. Equality $s(C) = Y$ cannot hold because in this situation $r(A) = \{*\}$ and $* \notin X$ implies $s(C) = Y \subseteq r(A) = \{*\}$, which is not possible for $X \neq \varnothing$. If $r(A) = A$ then $s(C) = Y \not\subset A = r(A)$. Possibility $r_C(A) = \{*\}$ cannot occur because $C \not\subset \{*\}$. So, relation $r_C(A) \neq \{*\}$ implies $r_C(A) = A$, situation that occurs if C = A. Therefore, the definition of s implies that there is $M' \subset M$ such that equality $A = \cap \{M \mid M \in M'\}$ holds, meaning that set A is convex with respect to M.□

3.2. Convexity with respect to a set of functions (K. Fan (1963))

Let X be a nonempty set and W be a given set of functions, $w: X \to R$. A set $A \subseteq X$ is said to be *convex with respect to* W (or, simply, W – *convex*) if for each point $x \notin A$ there is a function $w \in W$ such that $\sup w(A) < w(x)$.

I. Singer (1984) extends the notion of W–convexity, allowing functions of type $w: X \to \overline{\mathbb{R}} = [-\infty, +\infty]$. He proves that the W – convexity concept obtained in this manner is equivalent to the M – convexity. According to I. Singer (1997), denoting by

$$S_d(w) = \{x \in X \mid w(x) \leq d\},$$

then a set $A \subseteq X$ is convex with respect to W if and only if

$$A = \cap \{S_d(w) \mid (w, d) \in W \times \mathbf{R}, \sup w(A) \leq d\}.$$

It means that, for $M = \{S_d(w) \mid (w, d) \in W \times \mathbf{R}\}$, a set $A \subseteq X$ is W – convex if and only if $A = \cap \{M \mid M \in M'\}$ for

$$M' = \{S_d(w) \mid (w, d) \in W \times \mathbf{R}, \sup w(A) \leq d\} \subseteq M,$$

meaning that A is convex with respect to M. Therefore, the property of convexity with respect to a set of functions W is a partial (a, e) - ((S, s), R) convexity. Elements S, s and R are defined according to the pattern from example 3.1.

3.3. Convexity with respect to a pair (W, φ) (J. Serader (1975)) Let X and W be arbitrary nonempty sets and a function $\varphi: X \times W \to \overline{\mathbb{R}} = [-\infty, +\infty]$. J. J. Moreau (166-67) called function φ a coupling function. A set $A \subseteq X$ is said to be *convex with respect to the pair* (W, φ) (or, briefly, (W, φ) – *convex*) if for each $x \notin A$ there is an element $w \in W$ such that

$$\sup \{\varphi(g, w) \mid g \in A\} < \varphi(x, w).$$

I. Singer (1997) proved that a set $A \subseteq X$ is convex with respect to the pair (W, φ) if and only if

$$A = \cap \{S_d(\varphi(\bullet, w)) \mid (w, d) \in W \times \mathbf{R}, \sup \{\varphi(g, w) \mid g \in G\} \leq d\},$$

with $S_d(\varphi(\bullet, w)) = \{x \in X \mid \varphi(x, w) \leq d\}$.

As in the previous example, a set $A \subseteq X$ is (W, φ) – convex if and only if

$$A = \cap \{M \mid M \in M'\}$$

for $M = \{S_d(\varphi(\bullet, w)) \mid (w, d) \in W \times \mathbf{R}\}$, and

$$M' = \{S_d(\varphi(\bullet, w)) \mid (w, d) \in W \times \mathbf{R}, \sup \{\varphi(g, w) \mid g \in G\} \leq d\} \subseteq M.$$

It means that A is convex with respect to M. Therefore, the property of convexity with respect to a pair (W, φ) is a partial (a, e) - ((S, s), R) convexity. The elements S, s and R are defined as in example 3.1.

3.4. Holomorphic convexity (B.A. Fuks (1962))

The holomorphic convexity of a set $A \subseteq C^n$ is a W – convexity when W is the set of all functions that are holomorphic in $A \subseteq C^n$. Therefore, the holomorphic convexity is also a partial (a, e) - ((S, s), R) convexity.

3.5. Pseudoconvexity convexity (L. Hörmander (1989))

Let $A \subseteq C^n$ be an open set. We recall that a function $u: A \to [-\infty, +\infty]$ is called plurisubharmonic if u is semicontinuous from above and for arbitrary z and w from C^n, the function $\lambda \to u(z+\lambda w)$ is subharmonic in the part of C where it is defined. The set of all plurisubharmonic functions on A is denoted by P(A). If K is a compact subset of the open set $A \subseteq C^n$ then the P(A) – hull of K is defined by

$$\hat{K}^{P}_{A} = \{z \mid z \in A, u(z) \leq \sup_{K} u \text{ for all } u \in P(A)\}.$$

The open set $A \subseteq C^n$ is *pseudoconvex* if there is a continuous plurisubharmonic function u in A such that the set $A_c = \{z \mid z \in A, u(z) < c\}$ is relatively compact in A (i.e. is contained in a compact subset of A) for every $c \in \mathbf{R}$. This property holds if and only if K relatively compact in A implies $\hat{K}^{P}_{A}$ relatively compact in A.

The pseudoconvexity of the open set $A \subseteq C^n$ is another W–convexity. Here W=P(A). Therefore, the pseudoconvexity is also a partial (a, e) - ((S, s), R) convexity.

3.6. Evenly convex subsets of a locally convex space (W. Fenchel (1952))

The even convexity is the first intersectional approach of the convexity.

A subset M of a locally convex space X is said to be an open half-space if there is a linear functional $\varphi \in X^* \setminus \{0\}$ and $d \in \mathbf{R}$ such that $M = \{x \in X \mid \varphi(x) < d\}$.

A subset $A \subseteq X$ is said to be *evenly convex* if A is the intersection of a family of open half-spaces. Obviously, the even convexity is another M-convexity property, if we take

$$\mathcal{M} = \{M \subseteq X \mid M = \{x \in X \mid \varphi(x) < d\}, \varphi \in X^* \setminus \{0\}, d \in \mathbf{R}\}.$$

Therefore, the even convexity is a partial (a, e) - ((S, s), R) convexity.

4. The class of (e, a) – ((S, S), r) intersectional convexities

This type of convexity is a weak intersectional approach. Let us consider the nonempty set X, a nonempty set Y, a nonempty subset $S \subseteq 2^X$, a family S of functions $s: S \to 2^Y$ and a transformation $r: 2^X \to 2^Y$.

Definition 4.1. *A set* $A \subseteq X$ *is said to be* (e, a) - ((S, S), r) *convex if for each subset* $C \in S$, $C \subseteq A$, *there is a function* $s \in S$ *which satisfies the inclusion*

$$s(C) \subseteq r(A).$$

There is only one intersectional convexity of this type.

4.1. Projective convexity (J. de Groot, H. de Vries (1958), T. Bisztriczky (1987))

Let P^3 be the real projective three – space. For a plane $p \subseteq P^3$ we denote the affine restriction P_p^3. A subset A of P^3 is *convex* if there is a plane $p \subseteq P^3$ disjoint from A, such that A equals to the convex hull of A in the affine restriction P_p^3.

Let us take $X = Y = P^3$, S is the set of all parts of X and

$$S = \{s_p \mid s_p: S \to 2^X, s_p(A) = \text{convex hull of A in the affine restriction } P_p^3,$$
$$p \subseteq P^3 \text{ plane}\}.$$

Considering that r is the identity in 2^X, and using the monotony of each function belonging to S one can prove that the projective convexity is a (e, a) – ((S, S), *r*) convexity.

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