Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVIII(2002), Nr. 1, 53 - 58

Dedicated to Costică MUSTĂŢA on his 60th anniversary

SOME GENERALIZATION OF THE LERRAY-SCHAUDER PRINCIPLE

Andrei HORVAT-MARC

Abstract: We extend the Lerray - Schauder principle to the Mönch operator, more precisely we establish an existence result for nonself Mönch operator.

MSC: 47H10, 54H25
Keywords: fixed point theorem, continuation principle, Banach Space

1. Introduction

Generally, a fixed point theorem can be applied if the operator inward his definition domain. So, it is important to establish what conditions are require such that there exists at least one fixed point for nonself operator. Such conditions was extended in many way. One of them uses the retraction mapping principle. This technique was presented in [8], [4] and other paper. The other way uses the continuation principle. In this paper we replace the inward hypothesis with Lerray - Schauder boundary condition and using the continuation principle we give an existence result.

First, we recall an abstract continuation principles. Let X and Y be two sets and the subsets $A \subset X$, respectively $B \subset Y$. Consider a mapping $H : X \times [0,1] \to Y$ and a set $\mathcal{A}$ of function from X into $[0,1]$ which are constant on A. Suppose that constant function 0 and 1 belong to $\mathcal{A}$. Also, consider a function ν which is define at least on the following family of subsets of X

$$\left\{ H\left(\cdot, a\left(\cdot\right)\right)^{-1}\left(B\right); a \in \mathcal{A} \right\} \cup \emptyset.$$

The nature of the value of ν does not import. Denote

$$S = \{x \in X; H\left(x, \lambda\right) \in B \text{ for some } \lambda \in [0,1]\}$$

and $H_\lambda = H(\cdot, \lambda)$ for each $\lambda \in [0,1]$.

Theorem 1.1 [5] *Suppose that the following condition are satisfied:*

(i) *for each $a \in \mathcal{A}$, there exists $a^* \in \mathcal{A}$ such that*

$$a^*(x) = \begin{cases} a(x) & \text{for } x \in S, \\ 0 & \text{for x} \in A, \end{cases}$$

(ii) *the mapping $F = H_0$ satisfies*

$$\nu\left(H(\cdot, a(\cdot))^{-1}(B)\right) = \nu\left(F^{-1}(B)\right) \neq \nu(\emptyset), \tag{1}$$

for any $a \in \mathcal{A}$ with $H(\cdot, a(\cdot))|_A = F|_A$.
Then, there exists at least one $x \in X \backslash A$ a solution to $H_1(x) \in B$. Moreover, $F = H_1$ also satisfies (1) *and*

$$\nu\left(H_1^{-1}(B)\right) = \nu\left(H_0^{-1}(B)\right). \tag{2}$$

This result can be useful for applications which use methods of fixed point theory if the function $\nu : X \to \{0,1\}$ is defined by

$$\nu(C) = \begin{cases} 1 \text{ if } C \neq \emptyset, \\ 0 \text{ if } C = \emptyset. \end{cases} \tag{3}$$

Indeed, consider the fixed point problem

$$T(u) = u$$

where $T : X \to X$ is an operator and X is a set. Assume that $Y = X \times X$, $B = \{(u,u) : u \in X\}$ and $u_0 \in X$. Let $H : X \times [0,1] \to X \times X$ be the mapping define by

$$H(x, \lambda) = ((1-\lambda)u_0 + \lambda T(u), u).$$

Then $H_1(u) \in B$ is equivalent with $T(u) = u$. So, the fixed point theorem is in a form for which it can apply the abstract continuation principle. This means if we know that the problem $H_0(u) \in B$, $u \in X$ has a solution then the problem $H_1(u) \in B$, $u \in X$ has a solution, too.

In the next, for a bounded set U of a real Banach space X we denote by $\alpha_k(U)$ the Kuratowski measure of noncompactness and by $\overline{cv}(U)$ note the convex closer of U.

Definition 1.1 Let X be a Banach space and $Y \subset X$ be a subset of X. Assume that the operator $T : Y \to X$ is continuous and bounded.

(D1) the operator T is **completely continuous** if for any bounded $A \in Y$, the set $T(A)$ is relatively compact in X.

(D2) if there is a constant $q \in (0,1)$ such that

$$\alpha_k(T(A)) < q\alpha_k(A) \tag{4}$$

for any bounded $A \in Y$, then T is called a (q, α_k) **- contraction**.

(D3) the operator T is said to be α_k- **condensator** if

$$\alpha_k(T(A)) < \alpha_k(A) \tag{5}$$

for any bounded $A \in Y$ which is not relatively compact.

(D4) if T is such that

$$A \subset Y \quad \text{countable, not relatively compact implies } \alpha_k\left(T\left(A\right)\right) < \alpha_k\left(A\right) \tag{6}$$

then T is called **Daher operator**;

(D5) let $u_0 \in Y$ and suppose that T has the following property:

$$A \in Y \text{ countable, } A \subset \overline{cv}\left(\{u_0\} \cup T\left(A\right)\right) \text{ implies that } A \text{ is relatively compact} \tag{7}$$

then T is a **Mönch operator**.

2. The Lerray - Schauder Principle for Mönch operator

In this section using the Mönch Fixed Point Theorem we extend some fixed point theorem for the case when the operator does not inward his definition domain. Now, we recall the Mönch Fixed Point Theorem.

Theorem 2.1 [3] *Let X be a Banach space, $Y \subset X$ be a nonempty, closed and convex subset of X and $u_0 \in Y$. Assume that the continuous operator $T : Y \to Y$ has the following property:*

$$A \subset Y \quad \textit{countable, } A \subset \overline{cv}\left\{\{u_0\} \cup T\left(A\right)\ \right\} \textit{ implies } A \textit{ is relatively compact.} \tag{8}$$

Then T has a fixed point in Y.

The main result of this paper is the next theorem

Theorem 2.2. *Let X be a Banach space, $Y \subset X$ be a closed convex subset of X and $U \subset X$ be an open subset of Y. Let $u_0 \in intU$ and consider the operator $T : \overline{U} \to Y$. Assume that T is Mönch operator and satisfies the Lerray - Schauder boundary condition*

$$\left(1-\lambda\right)u_0 + \lambda T\left(u\right) \neq u, \tag{9}$$

for any $u \in \partial U$ and $\lambda \in [0,1]$. Then T has a fixed point in $\overline{U}$.

Proof: Let

$$S = \left\{u \in \overline{U} : \left(1-\lambda\right)u_0 + \lambda T\left(u\right) = u \text{ for some } \lambda \in [0,1]\right\}.$$

Hence the sets S and ∂U are closed and $S \cap \partial U = \emptyset$. By Uryshon Lemma results that there is a function $a \in C\left(\overline{U}\ ;\ [0,1]\right)$ such that

$$a(u) = \begin{cases} 0 & u \in \partial U, \\ 1 & u \in S. \end{cases}$$

Now, consider the sets $X = \overline{U}$, $Y = X \times X$, $A = \partial U$, $B = \left\{(u,u) \in Y : u \in \overline{U}\right\}$ and the mapping $H : \overline{U} \times [0,1] \to \overline{U} \times \overline{U}$ defined by

$$H(u,\lambda) = ((1-\lambda)\, u_0 + \lambda T(u), u)$$

and

$$\Gamma = \left\{a \in C\left(\overline{U};[0,1]\right) : a\,|_{\partial U} = \text{const}\right\}.$$

Because of abstract continuation principle we must prove that for any $a \in \Gamma$ such that

$$H(\cdot, a(\cdot))|_{\partial U} = H_0|_{\partial U} \tag{10}$$

there exists a solution for $H(u, a(u)) \in B$. The previous equality from above is equivalent to

$$a(u)\,(T(u) - u_0) = 0, \text{ for every } u \in \partial U.$$

Consider the set $K = \overline{cv}\left\{\{u_0\} \cup T\left(\overline{U}\right)\right\}$ and the mapping $F : K \to K$ defined by

$$F(u) = \begin{cases} (1 - a(u))\, u_0 + a(u)\, T(u), & u \in \overline{U}, \\ u_0, & u \in K \backslash \overline{U}. \end{cases}$$

Hence $F(u) = u_0$ for every $u \in \partial U$, it follows that F is a continuous operator. Let $C \subset K$, than

$$\overline{C} = \overline{cv}\left\{\{u_0\} \cup F(C)\right\} \subset \overline{cv}\left\{\{u_0\} \cup F\left(C \cap \overline{U}\right)\right\}.$$

Since T is a Mönch operator, we have $\overline{C \cap \overline{U}}$ compact. Then $\overline{T\left(C \cap \overline{U}\right)}$ is compact and by Mazur Lemma [3], follows that $\overline{cv}\left\{\{u_0\} \cup F\left(C \cap \overline{U}\right)\right\}$ is compact. Hence $\overline{C}$ is compact, therefor F is a Mönch operator. By Mönch fixed point theorem, there exists $u \in K$ such that $F(u) = u$. If $u \neq u_0$ than $u \in \overline{U}$. Hence

$$(1-\lambda)\, u_0 + \lambda T(u) = u \tag{11}$$

and this is equivalent to $u \in S$. Then the problem $H(u, a(u)) \in B$ has a solution $u \in S$. Now, by abstract continuation principle we can say that there exists $u \in U$ such that $T(u) = u$ ■.

One of the consequences of this theorem is the classical Lerray - Scahuder Principle:

Theorem 2.3. *Let X be a Banach space, $Y \subset X$ be a nonempty, closed convex subset of X and $U \subset Y$ be an open, bounded subset of Y. Let $u_0 \in intU$ and consider the operator $T : \overline{U} \to Y$. Assume that T is completely continuous and*

$$(1-\lambda)\,u_0 + \lambda T\,(u) \neq u \; for \; any \; u \in \partial U \; and \; \lambda \in [0,1]$$

Then T has a fixed point in U.

Proof: We must prove that a completely continuous operator is a Mönch operator. Since $T : Y \to X$ is completely continuous results $\alpha_k\,(T\,(A)) = 0$ for any A a bounded subset of Y. For any $\overline{A} \subset \overline{cv}\,\{\{u_0\} \cup T\,(A)\}$ we have

$$\alpha_k\left(\overline{A}\right) < \alpha_k\left(\overline{cv}\,\{\{u_0\} \cup T\,(A)\}\right) = \alpha_k\,(T\,(A)) = 0$$

Then $\overline{A}$ is compact ■.

The next result is an extension of the Sadovskij fixed point theorem [3]

Theorem 2.4. *Let X be a Banach space and $Y \subset X$ be a nonempty closed convex subset of X and $U \subset Y$ be an open, bounded subset of Y. Let $u_0 \in intU$ and consider the operator $T : \overline{U} \to Y$. Assume that T is (q, α_k) -contraction and*

$$(1-\lambda)\,u_0 + \lambda T\,(u) \neq u \; for \; any \; u \in \partial U \; and \; \lambda \in [0,1]\,.$$

Then T has a fixed point in $\overline{U}$.

Proof: We claim that a (q, α_k) -contraction is Mönch operator. Indeed, for any $A \subset U$ such that $\overline{A} \subset \overline{cv}\,(\{u_0\} \cup T\,(A))$ we have

$$\alpha_k\left(\overline{A}\right) < \alpha_k\left(\overline{cv}\,\{\{u_0\} \cup T\,(A)\}\right) = \alpha_k\,(T\,(A)) < q \cdot \alpha_k\,(A)\,.$$

Hence $\alpha_k\,(A)\,(1-q) < 0$. This implies $q > 1$ which is in contradiction with $q \in (0,1)$. So, $\alpha_k\,(A) = 0$ ■.

The Darbo fixed point theorem [3] is generalize in the next result.

Theorem 2.5. *Let X be a Banach space and $Y \subset X$ be a nonempty, closed convex subset of X and $U \subset Y$ be an open, bounded subset of Y. Let $u_0 \in intU$ and consider the operator $T : \overline{U} \to Y$. Assume that T is α_k -condensator and*

$$(1-\lambda)\,u_0 + \lambda T\,(u) \neq u \text{ for any } u \in \partial U \text{ and } \lambda \in [0,1]\,.$$

Then T has a fixed point in $\overline{U}$.

Proof: If we show that a α_k-condensator is Mönch operator than the theorem is proved. For any $A \subset U$ such that $\overline{A} \subset \overline{cv}\left(\{u_0\} \cup T\left(A\right)\right)$ we have

$$\alpha_k\left(\overline{A}\right) < \alpha_k\left(\overline{cv}\left\{\{u_0\} \cup T\left(A\right)\right\}\right) < \alpha_k\left(A\right)$$

Hence $\alpha_k\left(A\right) = 0$, and this is equivalent to assertion that $\bar{A}$ is compact. An analogue result it can establish for a Daher operators.

REFERENCES

[1] Altman M., *A Fixed Point Theorem in Hilbert Space*, Bull. Acad. Pol. Sc. **5**, 19-22 (1957).

[2] Altman M., *A Fixed Point Theorem in Banach Space*, Bull. Acad. Pol. Sc. **5**, 89-92 (1957).

[3] Guo D., Lakshmikantham V. & Liu X., *Nonlinear Integral Equation in Abstract Spaces*, Kluwer Academic Publishers,Dordrecht-Boston-London, 1996.

[4] Horvat-Marc A., *Retraction Methods in Fixed Point Theory*, Seminar on Fixed Point Theory Cluj Napoca, **1**, 39-54 (2000).

[5] Precup R., *Existence Theorems for Nonlinear Problems by Continuation Methods*, Nonlinear Analysis, Theory, Methods & Application, **6**, 3313-3322 (1997).

[6] Precup R., *Existence Criteria for Integral Equations in Banach Spaces*, J. of Inequal. & Appl., **6**, 77-97 (2001).

[7] Rus I.A., *Some Open Problems of Fixed Point Theory*, Seminar of fixed point structures, Cluj-Napoca, Preprint **3**, 19-39. (1999).

[8] RUS I.A., *The Fixed Point Structures and the Retraction Mappping Principles*, Proceedings oh the Conference on Differential Equation, Cluj-Napoca, November 21-23 (1985).

Received: 30.03.2002

North University of Baia Mare
Departament of the Mathematics and
Computer Science
Victoriei 76, 4800 Baia Mare, Romania
E-mail:hmand@personal.ro