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Dedicated to Costică MUSTĂŢA on his 60^{th} anniversary

THE EXISTENCE OF MAXIMAL ELEMENTS FOR A PAIR OF MULTIVALUED OPERATORS

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1. Introduction

The applications of the technique of the fixed point theory in the coincidence theory are well-known (see, for example, Browder [2], Rus [10], [11], Sessa-Mehta [12], Muntean-Petruşel [9] and others).

Also, the connection between the existence of the fixed points and the existence of the maximal elements of some multivalued operators has preoccupied numerous mathematicians (see Border [1], Mehta [4], [5], Muntean [7]).

Less known, and until now, less numerous are the abstract schemas to obtain new results upon the existence of maximal elements for a pair of multivalued operator, using the coincidence theorems. This direction was taken up in the last decade.

Indeed, G. Mehta and S. Sessa (see [6], [12]) established some coincidence theorems for upper semicontinuous multifunctions using the well-known Himmelberg's fixed point principle (see [3]) and as a consequence of these theorems, they proved results concerning the existence of maximal elements for a pair of multifunctions defined on subsets of topological vector spaces.

In [9] we obtain the lower semicontinuous version of the coincidence theorems due to Mehta and Sessa using, instead of Himmelberg's result, a new fixed point principle of X.Wu ([13]). This paper is a continuation of these investigations. Its purpose is to prove some existence theorems of maximal elements for a pair of lower semicontinuous multivalued operators defined on subsets of locally convex Hausdorff topological vector spaces using our coincidence theorems.

We follow here the technique used in Mehta and Sessa [6].

2. Preliminaries

We first recall some basic notions and notations, following mainly Rus [10].

Let E be a topological vector space and X be a nonempty subset of E. We use the well-known symbols:

$$\mathcal{P}(X) := \{A \mid A \subset X\} \quad \text{and} \quad P(X) := \mathcal{P}(X) \setminus \{\varnothing\}.$$

Then, we denote by $\mathcal{P}_{cv}(X)$ (resp. $P_{cv}(X)$) the collection of all convex subsets, possibly empty (resp. nonempty) of X.

As an important particular case, we can have the subcollection $\mathcal{P}_{cl,cv}(X)$ (resp. $P_{cl,cv}(X)$) of all closed convex subsets, possibly empty (resp. nonempty) of X.

If K is a nonempty subset of a topological vector space, then convK and IntK denote the convex hull, respectively the topological interior of K.

Given a multivalued operator $T : X \multimap Y$, we denote:

$$T(A) := \bigcup_{x \in A} T(x);$$

$$T^{+}(B) := \{x \in X \mid T(x) \subset B\};$$

$$T^{-}(B) := \{x \in X \mid T(x) \cap B \neq \varnothing\},$$

where $A \subset X$ and $B \subset Y$.

The inverse of T is the multivalued operator denoted by $T^{-1} : Y \multimap X$ and given by $T^{-1}(y) := \{x \in X \mid y \in T(x)\}$.

Definition 2.1. If X, Y are two topological space, then a multivalued operator $T : X \multimap Y$ is said to be

i) ***upper semicontinuous*** (in short, ***u.s.c.***), if for any open subset $A \subset Y$, the set $T^{+}(A)$ is open in X;

ii) ***lower semicontinuous*** (in short, ***l.s.c.***), if for any open subset $B \subset Y$, the set $T^{-}(A)$ is open in X.

Definition 2.2. If $T : X \multimap Y$ is a multivalued operator, then $\bar{x} \in X$ is a ***maximal element of T*** iff $T(\bar{x}) = \varnothing$.

Definition 2.3. If $T, S : X \multimap Y$ are multivalued operators, then $\bar{x} \in X$ is a ***coincidence point for T and S***, iff $T(\bar{x}) \cap G(\bar{x}) \neq \varnothing$.

Remark 2.4. In the next theorem, the multivalued operators T and S we will define as follows: $T : X \multimap Y$ and $S : Y \multimap X$. In this case, if there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{y})$ and $\bar{y} \in T(\bar{x})$, then because $\bar{y} \in S^{-1}(\bar{x})$, it results that $\bar{y} \in T(\bar{x}) \cap S^{-1}(\bar{x})$, i.e. $\bar{x}$ is a coincidence point for the pair T and S^{-1}.

Having introduced these basic notations and concepts, we can concentrate our attention to two auxiliary results.

We shall use in the main section the following coincidence theorem.

Theorem 2.5. (Muntean-Petrusel [9]). *Let X be a nonempty paracompact convex subset of a locally convex Hausdorff topological vector space E, D a nonempty set of a topological vector space Y. Let $S : D \to P(X)$ and $T : X \to P(D)$ two multivalued operators such that:*

a) S is l.s.c.;

b) $S(y) \in P_{cl,cv}(X)$, for each $y \in D$;

c) $Q(x)$=conv $T(x)$ is a subset of D;

d) $S(D) \subset C$, where C is a compact metrizable subset of X;

e) for each $x \in X$ there exists $y \in D$ such that $x \in IntQ^{-1}(y)$.

Then there exist $\bar{x} \in X$ and $\bar{y} \in D$ such that $\bar{x} \in S(\bar{y})$ and $\bar{y} \in Q(\bar{x})$.

Theorem 2.6. (Mehta-Sessa [6]). *Let X be a nonempty paracompact convex subset of a locally convex topological vector space E. Let D be a nonempty subset of a topological vector space Y, $S : D \to \mathcal{P}(X)$ be an u.s.c. multivalued operator and $T : X \to \mathcal{P}(D)$ be a multivalued operator such that:*

a) for each $(x, y) \in X \times D$, $x \in S(y)$ implies that $y \notin Q(x) = convT(x)$;

b) for each $y \in D$, $S(y) \in \mathcal{P}_{cl,cv}(X)$;

c) for each $x \in X$, $Q(x) \subset D$;

d) $S(D) \subset C$, where C is a compact subset of X;

e) for each $x \in X$ with $T(x) \neq \emptyset$, there exists a point $y \in D$ such that $x \in IntQ^{-1}(y)$.

Then either T has a maximal element in X or S has a maximal element in D.

3. Existence theorems of maximal elements

We now establish some results for the existence of maximal elements for a pair of multivalued operators. First we prove the following lower semi-continuous version of Theorem 2.6, using our previous result.

Theorem 3.1. *Let X be a nonempty paracompact convex subset of a locally convex Hausdorff topological vector space E, D be a nonempty subset of a topological vector space Y. Let $S : D \to \mathcal{P}(X)$ and $T : X \to \mathcal{P}(D)$ two multivalued operators such that:*

a) S is l.s.c.;

b) $S(y) \in \mathcal{P}_{cl,cv}(X)$, for each $y \in D$;

c) $Q(x) = convT(x) \subset D$, for each $x \in X$;

d) $x \in S(y)$ implies that $y \notin Q(x)$, for each $(x, y) \in X \times D$;

e) $S(D) \subset C$ where C is a compact metrizable subset of X;

f) for each $x \in X$ with $T(x) \neq \emptyset$, there exists $y \in D$ such that $x \in IntQ^{-1}(y)$.

Then either T has a maximal elements in X or S has a maximal element in D.

Proof. Suppose that $T(x) \neq \emptyset$ and $S(y) \neq \emptyset$, for each $x \in X, y \in D$. Then, by Theorem 2.5 gives a contradiction to the condition d). □

For simplicity, assume that $E = Y$ and $T(x)$ is convex for each $x \in X$, so that $T(x) = Q(x)$. Then, from Theorem 3.1 we get the following result.

Corollary 3.2. *Let X be a nonempty paracompact convex subset of a locally convex Hausdorff topological vector space E and let D be a nonempty subset of E.*

Suppose that $S : D \to \mathcal{P}(X)$ and $T : X \to \mathcal{P}(D)$ are two multivalued operators satisfying the following assertions:

a) S is l.s.c.;

b) $S(y) \in \mathcal{P}_{cl,cv}(X)$, for each $y \in D$;

c) $T(x) \in \mathcal{P}_{cv}(D)$, for each $x \in X$;

d) $x \in S(y)$ implies that $y \notin T(x)$, for each $(x,y) \in X \times D$;

e) $S(D) \subset C$ where C is a nonempty compact metrizable subset of X;

f) for each $x \in X$ with $T(x) \neq \varnothing$, there exists $y \in D$ such that $x \in IntT^{-1}(y)$.

Then either S has a maximal element in D or T has a maximal element in X.

Corollary 3.3. *Let X be a nonempty paracompact convex subset of a locally convex Hausdorff topological vector space E and let D be a compact convex and metrizable subset of X. If* $T : X \to \mathcal{P}(D)$ *is a multivalued operator such that:*

a) $x \notin Q(x) = convT(x)$, for each $x \in X$;

b) for each $x \in X$, with $T(x) \neq \varnothing$, there exists $y \in D$ such that $x \in IntQ^{-1}(y)$;

then T has a maximal element in X.

Proof. Since D is convex, we have $Q(x) \subset D$, for each $x \in X$. By taking $E = Y$, $D = C \subset X$ and $S(x) = \{x\}$, for each $x \in X$ in Theorem 3.1, we observe that all the conditions of this theorem are satisfied. Then, there exist elements $x' \in X$, $y' \in D$ such that either $S(y') = \varnothing$ or $T(x') = \varnothing$.

Because S is the identity operator, hence $S(y') = \{y'\}$, we obtain $T(x') = \varnothing$. □

Using a stronger condition instead of f) we immediately get the following result from Theorem 3.1:

Theorem 3.4. *Let X be a nonempty compact convex and metrizable subset of a locally convex Hausdorff topological vector space E, D a nonempty set of a topological vector space Y, and* $S : D \to \mathcal{P}(X)$, $T : X \to \mathcal{P}(D)$ *two multivalued operators satisfying:*

a) S is l.s.c.;

b) $S(y) \in \mathcal{P}_{cl,cv}(X)$, for each $y \in D$;

c) $T(x) \in \mathcal{P}_{cv}(D)$, for each $x \in X$;

d) $x \in S(y)$ implies that $y \notin T(x)$, for each $(x,y) \in X \times D$;

e) $T^{-1}(y)$ is an open subset in X, for each $y \in D$.

Then either T has a maximal element in X or S has a maximal element in D.

Remark 3.5. Condition d) of Corollary 3.2 may be interpreted in the following way: if we think of S and T as preference multivalued operators, the S(x) may be regarded as the upper set of the alternatives strictly preferred to x, i.e. $S(x) = \{y \in X \mid y \succ x\}$.

The interpretation of T(x) is similar.

Now condition d) says that S and T are not inverses of one another. In the special cases S=T, this condition becomes: $x \in T(y)$ implies that $y \notin T(x)$, which is the asimmetry condition for one multivalued operator.

The condition a) of Corollary 3.3 is the usual irreflexivity condition for one multivalued operator used in special literature.

At the end of this section we present a generalization of Theorem 2.6 and Theorem 3.1, which shows that for any coincidence theorem for a pair of multivalued operators we can attach an existence theorem of the maximal elements for that pair.

Theorem 3.6. *Let X be a nonempty subset of a topological vector space E and D be a nonempty subset of a topological vector space Y.*

If we suppose that $T : X \to \mathcal{P}(D)$, and $S : D \to \mathcal{P}(X)$ are two multivalued operators, which satisfy the hypotheses of a coincidence theorems, and in addition, they satisfy the following condition:

(c) *for each $(x, y) \in X \times D$, $x \in S(y)$ implies that $y \notin T(x)$, then either T has a maximal element in X or S has a maximal element in D.*

Proof. Suppose that $T(x) \neq \varnothing$ and $S(y) \neq \varnothing$, for each $x \in X, y \in D$. Then, in virtue of coincidence theorem there exist $\bar{x} \in X$ and $\bar{y} \in D$ such that $\bar{x} \in S(\bar{y})$ and $\bar{y} \in T(\bar{x})$, a contradiction to the condition (c). □

Remark 3.7. In the present paper, starting from Theorem 2.5, in the same manner, we got Theorem 3.1.

4. An economical interpretation

The finality of all these results is their application in mathematical economics and in the game theory to demonstrate the existence of some Nash equilibrium points.

The way of construction the operators S and T from the Theorem 2.5 and 3.1, 3.4 and 3.6, respectively from the Corollary 3.2, permit their interpretation as ***operators of response*** in an economy with "*neighbourhood effects*". That is, in any economy, some producers activity spreads positive or negative effects on the consumers or on other economic agents, withour being rewarded or charged. These are enlisted in the category ***of external*** (or ***neighbouring***) ***effects***. After the consequences they propagate, the external (of neighbouring) effects may be:

- ***positive external effects***, which appear as benefits for the consumers or the economic agents, others that those who generated them;

- ***negative external effects***, whose presence generates displeasures for the consumers and the economic agents who enter, inevitably, in touch with the economic agent who generated them.

In the actual economical system, extremely complex, these effects cannot be ignored, influencing in a considerable manner the optimization process of an economic agent's activities. In addition, they are very important for the study of the economic equilibrium.

We consider an economy $\mathcal{E}$ where two economic agents spread to each other "*neighbourhood effects*". If the dependence between the two agents is resiprocal, we then define ***the operators of response***, such as: $T : X \to P(Y)$ and $S : Y \to P(X)$.

In the same way we can recognize in the demonstrated theorems the Nash equilibrium of this economy, namely the point $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{y})$ and $\bar{y} \in T(\bar{x})$. As a consequence of this observation, there also results a method to demonstrate the existence of the Nash equilibrium of an economy, namely the stating of some coincidence theorems for a pair of multivalued operators.

In conclusion, the theorems above offer some existential results of the Nash equilibium for an abstract economy.

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