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*Dedicated to Costică MUSTĂŢA on his $60^{th}$ anniversary*

# ON THE CONVERGENCE OF THE SERIES $\sum a_n^{1-x_n/\log(1+n)}$

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**Abstract.** We show that, for any sequence $(a_n)$ of positive numbers and any bounded sequence $(x_n)$ of real numbers, the series $\sum a_n$ and $\sum a_n^{1-x_n/\log(1+n)}$ either both converge or both diverge.

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Throughout this paper, the letters $\mathbb{N}$ and $\mathbb{R}$ will stand for the sets of all natural and real numbers, respectively. We start with a natural generalization of an inequality in [1]

**Lemma.** If $a, x, \delta \in \mathbb{R}$ and $1 < n \in \mathbb{N}$ such that $0 < a \leq 1$ and $|x| \leq \delta \leq \log(n)$, then

$$a^{1-x/\log(n)} < (a + n^{-2})e^{2\delta}.$$

**Proof.** If $x < 0$, then $1 < 1 - x/\log(n)$. Hence, since $0 < a \leq 1$ and $0 \leq \delta$, it follows that

$$a^{1-x/\log(n)} \leq a \leq a\,e^{2\delta}.$$

Suppose now that $0 \leq x$. If $a < n^{-2}$, then since $0 \leq 1 - x/\log(n)$ and $x \leq \delta$ it is clear that

$$a^{1-x/\log(n)} \leq \left(n^{-2}\right)^{1-x/\log(n)} = n^{2x/\log(n)-2} = e^{2x}/n^2 \leq e^{2\delta}/n^2.$$

While, if $n^{-2} \leq a$, then $a^{-1/\log(n)} \leq \left(n^{-2}\right)^{-1/\log(n)} = n^{2/\log(n)} = e^2$. Hence, since $0 < a$ and $0 \leq x \leq \delta$, it follows that

$$a^{1-x/\log(n)} = a\left(a^{-1/\log(n)}\right)^x \leq a\left(e^2\right)^x = a\,e^{2x} \leq a\,e^{2\delta}.$$

Therefore, the required inequality is also true.

Now, analogously to the main result of [2], we can also prove the following

**Theorem.** *Let* $(a_n)$ *be a sequence in* $\mathbb{R}$ *such that* $a_n > 0$ *for all* $n \in \mathbb{N}$. *Then the following assertions are equivalent: (1) the series* $\sum a_n$ *converges; (2) the series* $\sum a_n^{1-x_n/\log(1+n)}$ *converges for all bounded sequence* $(x_n)$ *in* $\mathbb{R}$; *(3) the series* $\sum a_n^{1-x_n/\log(1+n)}$ *converges for some bounded sequence* $(x_n)$ *in* $\mathbb{R}$.

**Proof.** Suppose that the assertion (1) holds and $(x_n)$ is a bounded sequence in $\mathbb{R}$. Then $(a_n) \to 0$ and $\delta = \sup_{n\in\mathbb{N}} |x_n| < +\infty$. Therefore, there exists $n_0 \geq e^{\delta}$ such that $a_n \leq 1$ for all $n \geq n_0$. Now, by the above lemma, it is clear that

$$a_n^{1-x_n/\log(1+n)} < (a_n + (1+n)^{-2})\, e^{2\delta}$$

for all $n \geq n_0$. Hence, since the series $\sum a_n$ and $\sum (1+n)^{-2}$ converge, it follows that the series $\sum a_n^{1-x_n/\log(1+n)}$ also converges. Since the implication (2)(3) is trivially true, suppose now that the assertion (3) holds. Define $\delta = \sup_{n\in\mathbb{N}} |x_n|$ and choose $n_0 \in \mathbb{N}$ such that $e^{1+\delta} \leq n_0$. Then, for all $n \geq n_0$, we have

$$\begin{aligned} 1 &\leq \log(n_0) - \delta < \log(1+n) - \delta \leq \\ &\leq \log(1+n) - x_n \leq |x_n - \log(1+n)|. \end{aligned}$$

Therefore, we may define a sequence $(y_n)$ in $\mathbb{R}$ such that

$$y_n = x_n \log(1+n)/(x_n - \log(1+n))$$

for all $n \geq n_0$. Then, by the triangle inequality, it is clear that

$$\begin{aligned} |y_n| &= |x_n + x_n^2/(x_n - \log(1+n))| \leq \\ &\leq |x_n| + |x_n|^2/|x_n - \log(1+n)| \leq \delta + \delta^2 \end{aligned}$$

for all $n \geq n_0$. Therefore, the sequence $(y_n)$ is bounded. Hence, by the implication (1)(2), it follows that the series $\sum (a_n^{1-x_n/\log(1+n)})^{1-y_n/\log(1+n)}$ converges.

Now, since

$$a_n = (a_n^{1-x_n/\log(1+n)})^{1-y_n/\log(1+n)}$$

for all $n \geq n_0$, it is clear that the assertion (1) also holds.

The following example shows that the assumption that the sequence $(x_n)$ is bounded cannot be dropped or even cannot be weakened to the assumption that $(x_n/\log(1+n))$ is a null sequence.

**Example.** Let $(a_n)$ and $(x_n)$ be sequences in $\mathbb{R}$ such that $a_1 > 0$ and

$$a_n = \frac{1}{n\,(\log(n))^2} \quad \text{and} \quad x_n = \frac{\log(1+n)}{1+\sqrt{\log(n\log(n))}}$$

for all $n \geq 2$. Then the series $\sum a_n$ converges, but the series $\sum a_n^{1-x_n/\log(1+n)}$ diverges despite that $(x_n/\log(1+n)) \to 0$. By using Cauchy's condensation test, it can be easily shown that the series $\sum a_n$ converges, but the series $\sum a_n \log(n)$ diverges [3, p. 399]. Therefore, to prove the divergence of the series $\sum a_n^{1-x_n/\log(1+n)}$, it is enough to show only that

$$a_n \log(n) \leq a_n^{1-x_n/\log(1+n)}$$

for all $n \geq 3$. For this, assume that $n \geq 3$ and define

$$q_n = \sqrt{\log\big(n \log(n)\big)}\,.$$

Then, by using that $e \leq n$ and the functions log and sqrt are increasing, we can easily see that $1 \leq \log(n)$, $\log(n) \leq \log\big(n \log(n)\big)$, and hence $\sqrt{\log(n)} \leq q_n$. Hence, since $\log(x) \leq \sqrt{x}$ for all $x > 0$, we can infer that

$$\log\big(\log(n)\big) \leq \sqrt{\log(n)} \leq q_n\,.$$

This implies that $q_n \log\big(\log(n)\big) \leq q_n^2$. Therefore, we also have

$$\big(\log(n)\big)^{q_n} = e^{q_n \log(\log(n))} \leq e^{q_n^2} = e^{\log(n \log(n))} = n \log(n)\,.$$

This implies that $\big(\log(n)\big)^{1+q_n} \leq n\big(\log(n)\big)^2 = a_n^{-1}$.

Therefore, we also have

$$\log(n) \leq \big(a_n^{-1}\big)^{1/(1+q_n)} = a_n^{-1/(1+q_n)}.$$

Hence, it follows that

$$a_n \log(n) \leq a_n^{1-1/(1+q_n)} = a_n^{1-x_n/\log(1+n)}.$$

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