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Dedicated to Costică MUSTAŢA on his 60th *anniversary*

ON THE APPROXIMATION OF SOLUTIONS TO NONLINEAR OPERATORS BETWEEN METRIC SPACES

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Abstract. A Gauss-Seidel-type method for the solution of linear systems, based on the decomposition of the system matrix into four matrices blocks, hass been proposed by R. Varga in [3]. The convergence of this method was studied in [1] and [2].

In this paper we shall extend the ideas contained in the above quoted works to the case of nonlinear system equations.

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In the paper [3], R. Varga proposes a Gauss-Seidel type method for solving linear systems, which is based on decomposing the matrix of the system in four submatrix blocks. The convergence of this method has been in [1] and [2].

We shall extend these ideas to the case of nonlinear systems.

Let (X_i, ρ_i), $i = 1, 2$, two complete metric spaces, and $X = X_1 \times X_2$, $F : X \to X_1$, $G : X \to X_2$ two mappings. We are interested in studying the existence and uniqueness of the solution of the system

$$\begin{aligned} u &= F(u, v) \\ v &= G(u, v), (u, v \in X). \end{aligned} \tag{1}$$

In this sense, we shall consider the sequences $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ generated by the Gauss-Seidel method, i.e.,

$$\begin{aligned} u_{n+1} &= F(u_n, v_n) \\ v_{n+1} &= G(u_{n+1}, v_n), (u_0, v_0) \in X, n = 0, 1, 2, \ldots \end{aligned} \tag{2}$$

Let $D_i \subset X_i, i = 1, 2$ and $D = D_1 \times D_2$. We shall assume that F and G verify Lipschitz-type conditions on D, i.e., there exist $\alpha, \beta, a, b \geq 0$ such that

$$\begin{aligned}\rho_1\left(F\left(x_1, y_1\right), F\left(x_2, y_2\right)\right) &\leq \alpha \rho_1\left(x_1, x_2\right) + \beta \rho_2\left(y_1, y_2\right) \\ \rho_2\left(G\left(x_1, y_1\right), G\left(x_2, y_2\right)\right) &\leq a \rho_1\left(x_1, x_2\right) + b \rho_2\left(y_1, y_2\right)\end{aligned} \tag{3}$$

for all $(x_i, y_i) \subseteq D, i = 1, 2$.

For the study of the convergence of (2) we consider two sequences of real numbers $(f_n)_{n \geq 0}, (g_n)_{n \geq 0}$ with nonnegative terms, obeying the following system of difference inequalities:

$$\begin{cases} f_n \leq \alpha f_{n-1} + \beta g_{n-1} \\ g_n \leq a f_n + b g_{n-1}, \ n = 1, 2, \ldots, \end{cases} \tag{4}$$

where α, β, a, b are given in (3).

We associate to (4) the following system in the unknowns h, k

$$\begin{aligned}\alpha + \beta h &= hk \\ ah + b &= hk\end{aligned} \tag{5}$$

It was shown in [1] that if α, β, a, b obey

$$\begin{aligned}&\alpha + b + a\beta < 2 \\ &(1 - \alpha)(1 - b) - a\beta > 0 \\ &\alpha > 0, b > 0,\end{aligned} \tag{6}$$

then the system (5) has two real solutions (h_i, k_i), $i = 1, 2$ such that $0 < h_i k_i < 1$, $i = 1, 2$, and one of these solutions has both the components positive. Denote by (h_1, k_1) this solution, i.e., $h_1 > 0, k_1 > 0$, so that the elements of the sequences $(f_n)_{n \geq 0}$ and $(g_n)_{n \geq 0}$ obey

$$\begin{aligned}f_n &\leq C h_1^{n-1} k_1^{n-1} \\ g_n &\leq C h_1^{n} k_1^{n-1}, n = 1, 2, \ldots\end{aligned} \tag{7}$$

where $C = \max\left\{\alpha f_0 + \beta g_0, \left(a f_1 + b g_0\right)/h_1\right\}$.

Let $p_1 = h_1 k_1$ and $d_1 > 0$ be a positive number such that the sets

$$\begin{aligned}S_1 &= \left\{x \in X_1 \middle| \rho_1\left(x, u_0\right) \leq d_1/\left(1 - p_1\right)\right\}; \\ S_2 &= \left\{x \in X_2 \middle| \rho_2\left(x, v_0\right) \leq d_1 h_1/\left(1 - p_1\right)\right\},\end{aligned} \tag{8}$$

verify $S_i \subseteq D_i, i = 1, 2$.

Denoting $f_n = \rho_1\left(u_n, u_{n-1}\right), g_n = \rho_2\left(v_n, v_{n-1}\right), n = 1, 2, \ldots$ and taking into account the above relations we obtain the following result [2].

Theorem 1 *If the mappings F and G verify conditions (3) on the set D, $S_1 \times S_2 \subseteq D$, the numbers α, β, a, b verify (6) and $u_1 = F(u_0, v_0)$, $v_1 = G(u_1, v_0)$ are such that $\rho_1(u_1, u_0) \leq d_1$, $\rho_2(v_1, v_0) \leq d_1 h_1$, then the following statements hold:*

a) the sequences $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ converge, and denoting $\lim u_n = \overline{u}$, $\lim v_n = \overline{v}$, then $(\overline{u}, \overline{v})$ is the unique solution of (1) in the set $S = S_1 \times S_2$;

b) the following inequalities are true

$$(9) \qquad \begin{aligned} &\rho_1(\overline{u}, u_n) \leq \tfrac{d_1 p_1^n}{1-p_1} \\ &\rho_2(\overline{v}, v_n) \leq \tfrac{d_1 h_1 p_1^n}{1-p_1}, n = 0, 1, \ldots \end{aligned}$$

This theorem is proved using (3) and inequalities (4). We shall apply this Theorem to the study of a Gauss-Seidel type method for solving nonlinear operator equations.

Let (X, ρ) be a complete metric space and $X^m, X^s, X^{m-s}, 1 \leq s \leq m-1$ the chartesian products.

If $u, v \in X^i$, $i = \{m, s, m-s\}$, we define the metric in such a space in the following way: let $u = (u_1, \ldots, u_i)$, $v = (v_1, \ldots, v_i)$ and put

$$(10) \qquad \rho_i(u, v) = \max_{1\leq j\leq i} \{\rho(u_j, v_j)\}, i \in \{m, s, m-s\}.$$

Consider the mappings $\varphi_k : X^m \to X, k = \overline{1,m}$, the following system of equations:

$$(11) \qquad x_k = \varphi_k(x_1, x_2, \ldots, x_m), k = \overline{1,m}.$$

and define the mapping $\overline{F} : X^s \times X^{m-s} \to X^s$ resp. $\overline{G} : X^s \times X^{m-s} \to X^{m-s}$ in the following way. If $u = (u_1, \ldots, u_s) \in X^s$ and $v = (v_1, \ldots, v_{m-s}) \in X^{m-s}$ then

$$(12) \qquad \begin{aligned} &\overline{F}(u, v) = (\varphi_1(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s}), \ldots, \varphi_s(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s})), \\ &\overline{G}(u, v) = (\varphi_{s+1}(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s}), \ldots, \varphi_m(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s})). \end{aligned}$$

For solving of (12) we consider the following iterations

$$(13) \qquad \begin{aligned} &u_{n+1} = \overline{F}(u_n, v_n) \\ &v_{n+1} = \overline{G}(u_{n+1}, v_n), (u_0, v_0) \in X^s \times X^{m-s}, n = 0, 1, \ldots \end{aligned}$$

Assuming that the mappings $\varphi_k, k = \overline{1,m}$ verify the Lipschitz type conditions, i.e., to exist $a_{kl} \geq 0, k, l = \overline{1,m}$ such that $\forall (x_1, \ldots, x_m), (y_1, \ldots, y_m) \in D \subseteq X^m$ it follows

$$(14) \qquad \rho(\varphi_k(x_1, x_2, \ldots, x_m), \varphi_k(y_1, y_2, \ldots, y_m)) \leq \sum_{l=1}^{m} a_{kl} \rho(x_l, y_l), k = \overline{1,m}.$$

Denoting

$$\overline{\alpha} = \max_{1\le k\le s}\left\{\sum_{l=1}^{m} a_{kl}\right\} \quad \overline{\beta} = \max_{1\le k\le s}\left\{\sum_{l=s+1}^{m} a_{kl}\right\} \tag{15}$$

$$\overline{a} = \max_{s+1\le k\le m}\left\{\sum_{l=1}^{s} a_{kl}\right\} \quad \overline{b} = \max_{s+1\le k\le m}\left\{\sum_{l=s+1}^{m} a_{kl}\right\}$$

then it can be seen that the mappings $\overline{F}$ and $\overline{G}$ obey

$$\rho_s\left(\overline{F}(u,v), \overline{F}(x,y)\right) \le \overline{\alpha}\rho_s(u,x) + \overline{\beta}\rho_{m-s}(v,y)$$
$$\rho_{m-s}\left(\overline{G}(u,v), \overline{G}(x,y)\right) \le \overline{a}\rho_s(u,x) + \overline{b}\rho_{m-s}(v,y)$$

$\forall (u,v), (x,y) \in D = D^s \times D^{m-s}$. It is clear that if in Theorem 1 we set $X_1 = X^s, X_2 = X^{m-s}, \rho_1 = \rho_s, \rho_2 = \rho_{m-s}, \alpha = \overline{\alpha}, \beta = \overline{\beta}, a = \overline{a}, b = \overline{b}$ then $(u_0,v_0), (u_1,v_1)$ obey $\rho(u_0,u_1) \le d_1, \rho(v_0,v_1) \le d_1 h_1, \overline{S}_1 \subseteq D^s, \overline{S}_2 \subseteq D^{m-s}$, where

$$\overline{S}_1 = \left\{x \in X^s | \rho_s(x,u_0) \le d_1/(1-p_1)\right\},$$
$$\overline{S}_2 = \left\{x \in X^{m-s} | \rho_{m-s}(x,v_0) \le d_1 h_1/(1-p_1)\right\},$$

and assuming that the assumptions of Theorem 1 are satisfied, we get the same conclusions regarding the solution of (12).

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