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APPROXIMATIVE SOLUTION OF SOME DIFFERENTIAL EQUATIONS

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Abstract. In this paper we give some technique for approximative solution of some differential equation. The given method substituted some numerical method in the region where those methods are not very exact or we have not possibility to use those methods.

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Consider very simple example. We have the differential equation $y' = 100y$ and the initial value $y(0) = 1$. Using very known method Runge-Kutta (fourth-order method), with the step $h = 0,1$, we obtain $y(0,1) \approx 131,18$. The function $y(x) = e^{100x}$ is the particular solution of given initial value problem and $y(0,1) = e^{10} \approx 22026,465$. We can see that it is a very big difference. Using known formula for test of the step

$$\left|\frac{k_2 - k_3}{k_1 - k_2}\right| \leq 0,05$$

we obtain $h \leq 0,001$ for good result.

Using software MAPLE and method RK45 we obtain $e^{10} - y_{RK45} \approx 0,1147 \times 10^{38}$.

Consider the nonlinear system

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, $\mathbf{f} = (f_1, f_2, \dots, f_n)^T$. We now want to find the value $\mathbf{x}(t_0 + h)$. Supposing that $\mathbf{f}'_{\mathbf{x}}$ and $\mathbf{f}'_t$ are continuous in a given region, where

$$\tag{3}$$

$$\mathbf{f}'_{\mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}, \qquad \mathbf{f}'_t = \begin{pmatrix} \frac{\partial f_1}{\partial t} \\ \dots \\ \frac{\partial f_n}{\partial t} \end{pmatrix}. \tag{4}$$

Thus if we define $\mathbf{A} = \mathbf{f}'_{\mathbf{x}}(t_0, \mathbf{x}_0)$, then the system (1) can be written as

$$\mathbf{x}' = \mathbf{f}'_{\mathbf{x}}(t_0, \mathbf{x}_0)\mathbf{x} + \mathbf{h}(t) + [\mathbf{f}(t, \mathbf{x}) - \mathbf{h}(t) - \mathbf{f}'_{\mathbf{x}}(t_0, \mathbf{x}_0)\mathbf{x}], \tag{5}$$

where $\mathbf{h}(t) = \mathbf{f}'_t(t_0, \mathbf{x}_0)(t - t_0) + \mathbf{f}(t_0, \mathbf{x}_0) - \mathbf{f}'_{\mathbf{x}}(t_0, \mathbf{x}_0)\mathbf{x}_0$. Now let us define sequence $(\mathbf{x}^{(m)})_{m=0}^{\infty}$, where $\mathbf{x}^{(0)}$ is an approximation of the solution of the system

$$(\mathbf{x}^{(0)})' = \mathbf{A}\,\mathbf{x}^{(0)} + \mathbf{h}(t) \tag{6}$$

for $t \in \langle t_0, t_1 \rangle$, $t_1 = t_0 + h$ and $\mathbf{x}^{(t_0)} = \mathbf{x}(t_0)$. Now let's assume that $\mathbf{x}^{(m+1)}$ is the solution of the system

$$(\mathbf{x}^{(m+1)})' = \mathbf{A}\,\mathbf{x}^{(m)} + \mathbf{g}(t, \mathbf{x}^{(m)}), \tag{7}$$

where $\mathbf{x}^{(m)}(t_0) = \mathbf{x}(t_0)$, $i = 1, 2, \dots$. We have

$$\mathbf{x}^{(0)}(t) = e^{\mathbf{A}(t-t_0)}\,\mathbf{x}_0 + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)}\,\mathbf{h}(s)\,ds, \tag{8}$$

$$\mathbf{x}^{(m+1)}(t) = e^{\mathbf{A}(t-t_0)}\,\mathbf{x}_0 + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)}\,\mathbf{g}(s, \mathbf{x}^{(m)}(s))\,ds, \tag{9}$$

where

$$\mathbf{g}(s, \mathbf{x}(s)) = \mathbf{f}(s, \mathbf{x}(s)) - \mathbf{A}\mathbf{x}(s).$$

Technique of theorem Cayley-Hamilton (paper [3], or generalized form for example [14], [13]) to construct exponential of matrix $\mathbf{A}$ can be used .

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalue of $\mathbf{A}$, then

$$e^{\mathbf{A}t} = \alpha_0(t)\,\mathbf{E} + \alpha_1(t)\,\mathbf{A} + \ldots + \alpha_{n-1}(t)\,\mathbf{A}^{n-1}. \tag{10}$$

If $\lambda_i \neq \lambda_j$, for $i \neq j$, the functions $\alpha_i(t)$, $i = 1, 2, \ldots, n$ are solution of the system algebraic equations

$$\alpha_0(t) + \alpha_1(t)\,\lambda_i + \alpha_2\,\lambda_i^2 + \ldots + \alpha_{n-1}\,\lambda_i^{n-1} = e^{\lambda_i t},$$

where $i = 1, 2, \ldots, n$. If eigenvalue λ_r of $\mathbf{A}$ has multiplicity s, then we obtain s - equations corresponding the eigenvalue λ_r

$$\frac{d^j}{d\lambda_r^j}[\alpha_0(t) + \alpha_1(t)\,\lambda_i + \alpha_2\,\lambda_i^2 + \ldots + \alpha_{n-1}\,\lambda_i^{n-1}] = \frac{d^j}{d\lambda_r^j}[e^{\lambda_r t}].$$

As far as applications are concerned, the following Theorems is important.

Theorem 1 *If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of $\mathbf{A}$, where λ_j has multiplicity n_j and $n_1 + n_2 + \ldots + n_k = n$ and if a is any number larger than the real part of $\lambda_1, \lambda_2, \ldots, \lambda_k$, that is*

$$a > \max_{j=1,\ldots,k}(\Re\lambda_j)$$

then there exists a constant $N > 0$ such that

$$\| e^{\mathbf{A}t} \| \leq N e^{at} \quad (0 \leq t < \infty).$$

Theorem 2 *Let $u(t)$ be positive continuous function satisfying the inequality*

$$u(t) \leq r(t) + \alpha(t)\int_{t_0}^{t} \beta(\tau)u(\tau)\,d\tau, \quad t \geq t_0,$$

where $r(t)$, $\alpha(t), \beta(t)$ are continuous nonnegative function. Then

$$u(t) \leq r(t) + \alpha(t)\int_{t_0}^{t} r(\tau)\beta(\tau)e^{\int_\tau^t \alpha(s)\beta(s)\,ds}\,d\tau$$

for $t \geq 0$.

Theorem 3 *Let the function* $\mathbf{g}$ *satisfy the condition*

$$\| \mathbf{g}(t,\mathbf{u}) - \mathbf{g}(t,\mathbf{v}) \| \leq \beta(t) \| \mathbf{u} - \mathbf{v} \|, \tag{11}$$

where β *is positive continuous function on the interval* (t_0, t).

Then the solution $\mathbf{x}$ *of the system*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t,\mathbf{x}), \tag{12}$$

satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, *is bounded and valid*

$$\| \mathbf{x}(t)e^{-\mathbf{A}(t-t_0)} - \mathbf{x}_0 \| \leq r(t) + \alpha(t) \int_{t_0}^{t} r(\tau)\beta(\tau) e^{\int_{\tau}^{t} \beta(s)\,ds} d\tau = \Psi(t), \tag{13}$$

where

$$r(t) = \int_{t_0}^{t} \| e^{-\mathbf{A}(\tau-t_0)} \mathbf{g}(\tau, \mathbf{x}_0 e^{\mathbf{A}(\tau-t_0)}) \| d\tau.$$

Proof. Using the variation of constants formula on the (12), with $e^{\mathbf{A}t}$ as a fundamental matrix of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ we have

$$\mathbf{x}(t)e^{-\mathbf{A}(t-t_0)} - \mathbf{x}_0 = \int_{t_0}^{t} e^{-\mathbf{A}(\tau-t_0)} [\mathbf{g}(\tau, \mathbf{x}(\tau)) - \mathbf{g}(\tau, \mathbf{x}_0 e^{\mathbf{A}(\tau-t_0)}) + \mathbf{g}(\tau, \mathbf{x}_0 e^{\mathbf{A}(\tau-t_0)})] d\tau.$$

From this, by (11) we obtain

$$\| \mathbf{x}(t)e^{-\mathbf{A}(t-t_0)} - \mathbf{x}_0 \| \leq r(t) + \int_{t_0}^{t} \beta(\tau) \| \mathbf{x}(\tau)e^{-\mathbf{A}(\tau-t_0)} - \mathbf{x}_0 \| d\tau.$$

Using Theorem 2 we obtain (13).

Theorem 4 *Let* $r = \max_{(t_0,t_1)} r(t)$, $L = \max_{(t_0,t_1)} \beta(t)$ *and* $M = \max_{(t_0,t_1)} \| \mathbf{x}^{(m)}(t) - \mathbf{x}^{(m-1)}(t) \|$, $i = 1, 2, \ldots$ *Let real positive constants* a, N *exist such that* $\| e^{\mathbf{A}(t)} \| \leq N e^{at}$ *and such that*

$$\frac{NL}{a} \leq q < 1 \tag{14}$$

be satisfies on the interval $\langle t_0, t_1\rangle$.

Then the sequence of functions $(\mathbf{x}^{(m)}(t))_{m=0}^{\infty}$ *converges on the interval* $\langle t_0, t_1\rangle$ *to a solution* $\mathbf{x}(t)$, *of the integral equation*

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + e^{\mathbf{A}(t-t_0)}\int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)}\,\mathbf{g}(s, \mathbf{x}^{(m)}(s))\,ds \qquad (15)$$

and m-th approximation $\mathbf{x}^{(m)}(t)$ *satisfies the estimate*

$$\| \mathbf{x}(t) - \mathbf{x}^{(m)}(t) \| \leq \frac{LMN}{a + LM}(e^{(a+LM)(t_1-t_0)} - 1) \qquad (16)$$

for every $t \in \langle t_0, t_1\rangle$.

Proof. Using the conditions of theorem, we have

$$\| \mathbf{x}^{(m+1)}(t) - \mathbf{x}^{(m)}(t) \| \leq \int_{t_0}^{t} \| e^{\mathbf{A}(t-s)}[\mathbf{g}(s, \mathbf{x}^{(m)}(s)) - \mathbf{g}(s, \mathbf{x}^{(m-1)}(s))] \|\, ds \leq$$

$$\leq \int_{t_0}^{t} N e^{a(t-s)} L \| \mathbf{x}^{(m)}(s) - \mathbf{x}^{(m-1)}(s) \|\, ds \leq LNM \int_{t_0}^{t} e^{a(t-s)}\,ds.$$

Since

$$\| \mathbf{x}^{(p+m)}(s) - \mathbf{x}^{(m)}(s) \| \leq \| \mathbf{x}^{(p+m)}(t) - \mathbf{x}^{(p+m-1)}(t) \| + \ldots +$$

$$+ \| \mathbf{x}^{(m+2)}(t) - \mathbf{x}^{(m+1)}(t) \| + \ldots \| \mathbf{x}^{(m+1)}(t) - \mathbf{x}^{(m)}(t) \| \leq \qquad (17)$$

$$\leq \frac{LMN}{a}[\sum_{k=0}^{p-1} U_k - \sum_{k=0}^{p-1} s_k],$$

where

$$U_k = S_1 s_{p-1} + S_2 s_{p-2} + \ldots + S_{p-2} s_2 + S_{p-1} s_1,$$

$$S_t = e^{a(t-t_0)} \sum_{k=0}^{t} \frac{(LMt)^k}{k!}, \quad s_t = \sum_{k=0}^{t} (-\frac{LM}{a})^k, \quad \lim_{p\to\infty} S_p = S, \quad \lim_{p\to\infty} s_p = s.$$

The absolute convergence of the series $\sum_{k=0}^{\infty}(-\frac{LM}{a})^k$ follows from (14). By the D'Alembert test the series $\sum_{k=0}^{\infty}\frac{(LMt)^k}{k!}$ absolute converges on the interval $\langle t_0, t_1\rangle$.

In view of the definition $(\| \mathbf{x}^{(m+1)}(t) - \mathbf{x}^{m}(t) \|)_{m=0}^{\infty}$, this implies the absolute (and uniform) convergence on the interval $\langle t_0, t_1 \rangle$ of the series $(\| \mathbf{x}^{(m+1)}(t) - \mathbf{x}^{m}(t) \|)_{m=0}^{\infty}$. This also proves the convergence of the sequence $(\mathbf{x}^{(m)}(t))_{m=0}^{\infty}$. We have

$$\lim_{p\to\infty} \| \mathbf{x}^{(p+m)}(s) - \mathbf{x}^{(m)}(s) \| \leq \lim_{p\to\infty} \frac{LMN}{a}[\sum_{k=0}^{p+1} U_k - \sum_{k=0}^{p-1} s_k] = \frac{LMN}{a}[Ss - s].$$

This establishes (16).

We now wish to show that the limit function $\mathbf{x}(t)$ satisfies the integral equation (15). Using of the continuity of the function $\mathbf{g}$ we have

$$\mathbf{x}(t) = \lim_{m\to\infty} \mathbf{x}^{(m+1)}(t) = \lim_{m\to\infty} (e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)} \mathbf{g}(s, \mathbf{x}^{(m)}(s))\, ds) =$$

$$= e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)} \mathbf{g}(s, \lim_{m\to\infty} \mathbf{x}^{(m)}(s))\, ds =$$

$$= e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + e^{\mathbf{A}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{A}(s-t_0)} \mathbf{g}(s, \mathbf{x}(s))\, ds.$$

Conversely, if $\mathbf{x}(t)$ is a solution of (15) on some interval J containing t_0, then $\mathbf{x}(t)$ satisfies (1) on J and also the initial condition (2).

Given method can be used for approximative solution of the system of the differential equations represented by recurrent neuron networks [8], [15], [12].

$$x_i' = a_{ii}x_i + \sum_{j=1}^{n} b_{ij} g(x_j) + k_i, \tag{18}$$

where $i = 1, \ldots, n$ a_{ii} are negative real constants, b_{ij} positive real constants and k_i are real constants. The function g is nonlinear, monotonic, continuous and boudedness. The papers [5], [7], [8], [10], deal with some results for $g : \mathbf{R} \to \mathbf{R}$ and

$$g(u) = \frac{1}{1 + e^{-Gu}}, \tag{19}$$

where G is positive real constant.

If $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is an equilibrium point and condition

$$|a_{ii} - \lambda| \leq \sum_{j=1, i \neq j}^{n} |h_{ij} g'(X_j)|$$

is satisfied, then by method of Gerschgorin and method of first approximation are $Re\lambda_i < 0$.

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