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Dedicated to Costică MUSTAŢA on his 60^{th} anniversary

ON THE STABILITY OF THE FINITE ELEMENT MIXED APPROXIMATION FOR CONTACT PROBLEMS WITH FRICTION

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Abstract. The aim of the paper is to translate the contact conditions and the stability conditions which include an inf – sup condition for the mixed element method of the contact problems with friction, on the Lagrange multipliers space and on the space of the traces functions called the mortar space.

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1. Introduction

The paper study approximation of the unilateral contact problem with Coulomb law friction by mixed finite element method and study of the stability conditions using the numerical inf – sup condition. This problem is solved with the mortar method and the space Lagrange multipliers (see[1]) for the numerical contact problems with friction, by used finite element method.

2. Classical and variational formulation

Let $\Omega^\alpha \subset \mathbb{R}^d$, $\alpha = 1, 2$, $d = 2$ or 3, the polygonal domains occupied by two linear elastic bodies that come into contact with friction. Let denote by Γ^α his boundary $\partial\Omega^\alpha$ and let Γ_1^α, Γ_2^α and Γ_3^α be six open part of Γ^α, such that $\Gamma^\alpha = \bar{\Gamma}_1^\alpha \cup \bar{\Gamma}_2^\alpha \cup \Gamma_3^\alpha$, $\bar{\Gamma}_1 \cap \Gamma_3 = \emptyset$, $\Gamma_1^\alpha \cap \Gamma_2^\alpha = \emptyset$, and mes $(\Gamma_1^\alpha) > 0$. Both bodies share a common portion $\Gamma_C = \Gamma_3^1 = \Gamma_3^2$. Γ_C is the parametering of the two contact boundaries Γ_3^1 and Γ_3^2 (see [7]).

The boundary contact Γ_C is the candidate contact surface, meaning that the effective contact zone is contained in Γ_C during the deforming process.

We denote by $u^\alpha = (u^\alpha, \ldots, u^\alpha_d)$, the displacement field, $\varepsilon^\alpha = (\varepsilon^\alpha_{ij}(u^\alpha)) = \left(\frac{1}{2}(u^\alpha_{i,j} + u^\alpha_{j,i})\right)$ the strain tensor, and $\sigma = (\sigma_{ij}(u)) = (a^\alpha_{ijkl}\varepsilon^\alpha_{kl}(u^\alpha))$ the stress tensor with the usual summation convention, where $i, j, k, l = 1, \ldots, d$. For the normal and tangential components of the displacements vector and stress vector, we adopt the following notation: $u^\alpha_N = u^\alpha_i \cdot n^\alpha_i$, $u^\alpha_T = u^\alpha - u^\alpha_N \cdot n^\alpha$, $\sigma^\alpha_N = \sigma^\alpha_{ij} n^\alpha_i n^\alpha_j$, $(\sigma^\alpha_T)_i = \sigma^\alpha_{ij} n^\alpha_j - \sigma^\alpha_N n^\alpha_i$, where $n^\alpha = (n^\alpha_i)$ is the outward unit normal vector to $\partial\Omega^\alpha$.

Let us denote by f^α and h^α the density of body forces and traction forces, respectively. We assume that $a^\alpha_{ijkl} \in L^\infty(\Omega^\alpha)$, $1 \le i, j, k, l$, with the usual conditions of symmetry and elipticity.

The classical problem of the static unilateral contact problem is as follows: Find $u^\alpha = u^\alpha(x)$, s.t.:

$$div\, \sigma^\alpha(u^\alpha) = f^\alpha \quad \text{in} \quad \Omega^\alpha \tag{2.1}$$

$$\sigma^\alpha_{ij}(u^\alpha) = a^\alpha_{ijkl}\varepsilon^\alpha_{ij}(u^\alpha)\,\varepsilon^\alpha_{kl}(u^\alpha) \tag{2.2}$$

$$u^\alpha = 0 \quad \text{an} \quad \Gamma^\alpha_1 \tag{2.3}$$

$$\sigma^\alpha \cdot n^\alpha = h^\alpha \quad \text{on} \quad \Gamma^\alpha_2 \tag{2.4}$$

$$u^\alpha_N \le 0,\ \sigma^\alpha_N(u^\alpha) \le 0,\ u^\alpha_N \sigma^\alpha_N(u^\alpha) = 0 \quad \text{or} \quad \Gamma^\alpha_C \tag{2.5}$$

$$|\sigma^\alpha_T| \le -\mu\sigma^\alpha_N \quad \text{on} \quad \Gamma^\alpha_C \quad \text{and} \quad \begin{cases} |\sigma^\alpha_T| < \mu\sigma^\alpha_N \Rightarrow & u^\alpha_T = 0 \\ |\sigma^\alpha_T| = -\mu\sigma^\alpha_N \Rightarrow & \exists\, \lambda \ge 0,\, u^\alpha_T = -\lambda\sigma^\alpha_T \end{cases} \tag{2.6}$$

The friction coefficient μ is assumed to belong to $L^\infty(\Gamma_C)$ and to the set of Lagrange multipliers of $H^{1/2}(\Gamma_C)$ denoted by $M := \left(H^{1/2}_{00}(\Gamma_C)\right)'$, where $\left(H^{1/2}_{00}(\Gamma_C)\right)'$ is the dual space of the $H^{1/2}_{00}(\Gamma_C)$, this space is defined as the set of the restriction to Γ_C of the functions of $H^{1/2}(\Gamma_C)$ that vanish on $\partial\Omega^\alpha \setminus \Gamma_C$.

Therefore, the mapping $H^{1/2}(\Gamma_C) \ni v \longrightarrow \mu v \in H^{1/2}(\Gamma_C)$ is bounded by the norm $\|\mu\|$. We define $V = V^1 \times V^2$, and K by

$$V^\alpha = \{v^\alpha \in [H^1(\Omega^\alpha)]^d;\ v^\alpha = 0\, a.e. \text{ on } \Gamma^\alpha_1\},$$

$$K = \{v^\alpha \in V;\ g(v) := v^1_N - v^2_N \le 0\ a.e \text{ on } \Gamma_C\}.$$

The duality pairing on $\left[H^{1/2}(\Gamma_C)\right]^d, \left[H^{-1/2}(\Gamma_C)\right]^d$ is denoted by $<\cdot,\cdot>$. With the space M and with the convex cone $K_\Lambda = \{\mu \in M; \mu \ge 0\}$ it is possible to impose the condition $v \in K$ by means a suitable Lagrange multipliers on Γ_C. It is known that a variational formulations of the problem (2.1) – (2.6) is equivalent to the following mixed variational formulation (see [2]).

Find $u \in V$ and $\lambda \in K_\Lambda$ s.t.

$$a(u,v) + j(\lambda, v) \geq (L, v), \quad \forall\, v \in V \tag{2.7}$$

$$j(\lambda - \mu, v) \geq 0 \quad \forall\, \mu \in K_\Lambda \tag{2.8}$$

where $a(u,v) = \sum_{\alpha=1}^{2} \int_{\Omega^\alpha} a^\alpha_{ijkl} \varepsilon^\alpha_{ij}(u^\alpha)\, \varepsilon^\alpha_{kl}(v^\alpha) dx + \int_{\Gamma_C} c_N g(v)_+^{m_N} v_N ds$, a bilinear form continuous and coercive, and

$$j(\lambda, v) =< \lambda,\, g(v) >= \int_{\Gamma_C} \lambda c_T g(v)_+^{m_T} |v_T^2 - v_T^1| ds$$

where c_N, m_N, c_T, m_T are constants depending on interfaces proprieties, $b_+ = max(0, b)$ and $\mu = c_T/c_N$ is the coefficient of friction. The duality pairing between $H_{00}^{1/2}(\Gamma_C)$ and its dual spaces M is $< \cdot, \cdot >$, and

$$(L, v) = \sum_{\alpha=1}^{2} \left(\int_{\Omega^\alpha} f^\alpha v^\alpha\, dx + \int_{\Gamma_2^\alpha} h^\alpha\, v^\alpha ds \right)$$

is external work.

The choice the space M of Lagrange multiplier is essential in order to have the well - posedness of (2.7) – (2.8). With this chois ensures ensures that the continuous inf – sup condition holds: there exist a $\beta > 0$ s.t.:

$$\sup_{v \in V \setminus \{0\}} \frac{j(\mu, u)}{\|v\|_V} \geq \beta \|\mu\|_{M'}, \quad \forall\, \mu \in M \tag{2.9}$$

In the next section we will check whether the inf – sup conditions is satisfied on the discret space V_h and M_h.

3. Stability of the finite element mixed approximation

We consider the question of the nonconforming finite element approximation of the problem (2.7) – (2.8) using the mortar process.

Given a discretization parameter $h = (h_1, h_2)$, Ω^α is assumed polygonal and broken up into triangular elements, for $\Omega^\alpha \subset \mathbb{R}^2$. Then mesh T_h^α is regular, and on the Γ_C will be two different one - dimensional meshes $T_h^{\alpha,c}$, it is the set of all edges of $l \in T_h^\alpha$ on Γ_C. The finite element space used on Ω^α is then:

$$V_h = V_h^1 \times V_h^2, \text{ where } V_h^\alpha = \{v \in V^\alpha,\ v|_T \in P_k(T),\ \forall T \in T_h^\alpha\}$$

Let $W_h^\alpha(\Gamma_C)$ be the range of V_h^α by the normal trace operator on Γ_C

$$W_h^\alpha(\Gamma_C) = \{\varphi_h = v_h^\alpha|_{\Gamma_C} \cdot n^\alpha;\ v_h^\alpha \in V_h^\alpha\} \tag{3.1}$$

which is the mortar discretized space.

The space of the Lagrange multipliers (see [1]), will be used to express the contact conditions in a weak sense

$$M_h(\Gamma_C) = \{\psi_h \in W_h^\alpha(\Gamma_C),\ \psi_h|_T \in P_0(T),\ \forall T \in T_h^{\alpha,c},\ c_1, c_2 \in T\}$$

where c_1 and c_2 are the extreme points Γ_C and $\alpha = 1$ or 2. The mortar projection π_h^α on $W_h^\alpha(\Gamma_C)$ is defined for any function $\psi \in C(\Gamma_C)$ as follows:

$$(\pi_h^\alpha \varphi)(c_i) = \varphi(c_i), \quad i = 1 \text{ and } 2$$

$$\int_{\Gamma_C} (\varphi - \pi_h^\alpha \varphi)\psi_h \, ds = 0, \quad \forall \psi_h \in M_h(\Gamma_C).$$

The closed convex cone is

$$K_h = \{\mu_h \in M_h, \quad \mu_h \geq 0 \text{ on } \Gamma_C\}. \tag{3.2}$$

We are going to show now the inf – sup condition (2.9) is translated on the discreet spaces: the mortar space W_h and the Lagrange multipliers M_h. For this will be based on the Fortin Theorem [3], recalled in the following theorem:

Theorem 1. *Let V and M be Hilbert spaces, and let j be a bilinear continuous form on $V \times M$ such that the continuous inf – sup condition (2.9) is satisfied. Assume that we are given a family of subspaces $V_h \subset V$ and $M_h \subset M$, where h is a parameter spaning that, for each h, we are given a linear operator π_h from V to V_h with following properties:*

$$j(\mu_h,\ v - \pi_h v) = 0, \quad \mu_h \in M_h \tag{3.3}$$

and there exists a constant c_1, independent of h, such that

$$\|\pi_h v\|_V \leq c_1 \|v\|_V, \quad \forall v \in V \tag{3.4}$$

Then the discrete inf – sup condition

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{j(\mu_h, v_h)}{\|v_h\|_V} \geq \beta_1 \|\mu\|_M, \ \forall \mu_h \in M_h \tag{3.5}$$

holds with $\beta_1 = \beta/c_1$.

The proof can be found in [3, 4, 5].

We show in next lemma, that condition (3.5) is verified for an particular case; this case is:

$$\sup_{v_h \in V_h^2 \setminus \{0\}} \frac{j(\mu_h, v_h)}{\|v_h\|_{V^2}} \geq \beta \|\mu\|_M , \quad \forall \mu_h \in M_h \tag{3.6}$$

Lemma 1. *Assume that the continuous version of (3.6) holds, and exists a $\beta_2 > 0$ s.t.*

$$\sup_{v \in V^2 \setminus \{0\}} \frac{j(\mu, v)}{\|v\|_{V^2}} \geq \beta_2 \|\mu\|_M , \quad \forall \mu \in M \tag{3.7}$$

Assume moreover that for $h \in [0, h_0]$ there exist a linear operator π_η^2 from V^2 into V_h^2 satisfying

$$\int_{\Gamma_C} (v - \pi_h^2 v) \mu_h \, ds = 0 ; \quad \forall \mu_h \in M_h \tag{3.8}$$

and exists a constant c_1, independent of h, s.t.

$$\|\pi_h^2 v\|_{V^2} \leq c_1 \|v\|_{V^2} , \quad \forall v \in V^2 \tag{3.9}$$

Then (3.6) holds.

We assume that the decomposition cast on Γ_C by $\mathcal{T}_h^2$ coincides with decomposition give in $M_h(\Gamma_C)$. This is a rather particular case, but realistic and assume that the decomposition $\mathcal{T}_\eta^2$ is quasi − uniform. Under this assumption it is rather easy to check (see [8]and [6]) that for every $v_h \in V_h^2$ we can find $\bar{v}_h \in V_h^2$ s.t.

$$\bar{v}_h = v_h \quad \text{on} \quad \Gamma_C \tag{3.10}$$

and

$$\|\bar{v}_h\|_{V^2} \leq c \|v_h\|_{H_{00}^{1/2}(\Gamma_C)} \tag{3.11}$$

Under the above assumption we have following theorem.

Theorem 2. *Let W_h be the space of the trace of V_h^2 an Γ_C, and assume that we are given, for each h, an operator Π_h from $H_{00}^{1/2}(\Gamma_C)$ into W_h with the following proprieties:*

$$\int_{\Gamma_C} (w - \Pi_h w) \mu_h \, ds = 0 ; \quad \forall \mu_h \in M_h , \tag{3.12}$$

$$\|\Pi_h w\|_{H_{00}^{1/2}(\Gamma_C)} \leq c_2 \|w\|_{H_{00}^{1/2}(\Gamma_C)} , \quad \forall w \in H_{00(\Gamma_C)}^{1/2} . \tag{3.13}$$

where c_2 is a constant independent of h and v. Then an operator π_h^2 satisfying (3.8) and (3.9) exists and hence the inf − sup condition (3.6) holds.

Proof Let $v \in V^2$ we consider $w := v/_{\Gamma_C}$ and $w_h := \Pi_h w$. We then lift w_h in an arbitrary way, to an element $v_h \in V_h^2$ such that $v_h = w_h$ on Γ_C. Then we define $\pi_h^2 v$ as $\bar{v}_h$, and using relations (3.10) and (3.12) we obtained (3.8). Then (3.13) and (3.11) easily give (3.9).

Conclusions

The role of the Theorem 3.3 is to reduce the proof (3.6) to a property that depends only on M_h and W_h.

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