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Dedicated to Costică MUSTAŢA on his 60^{th} anniversary

FIXED POINTS FOR NON-SURJECTIVE EXPANSION MAPPINGS SATISFYING AN IMPLICIT RELATION

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Abstract. A general fixed point theorem for four non-surjective expansion mappings satisfying an implicit relation which generalize Theorem 4.1 of [7] is proved.

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1. Introduction.

Let (X, τ) be a topological space and $d : X \times Y \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = 0$. X is said to be d-complete if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ implies that the sequence $\{x_n\}$ is convergent in (X, τ). Complete metric space and complete quasi-metric space are examples of d-complete topological spaces. Recently, Hicks and Rhoades [3], Saliga [6], Sharma, Sahu, Bounias and Bonaly [7] proved several fixed point theorems in d-complete topological spaces. Let $T : X \to X$ be a mapping, T is w-continuous at x if $x_n \to x$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition [1]. Let S and T be mappings from a topological space (X, τ) into itself. The mappings S and T are said to be semi-compatible if the hold the following conditions:
(D_1) : $Sp = Tp$ for some $p \in X$ implies $STp = TSp$,
(D_2) : the w-continuity of T at a point p in X implies $\lim STx_n = Tp$, whenever x_n is a sequence in X such that $\lim Sx_n = \lim Tx_n = p$ for some p in X.
The following familly of real functions was introduced in [6]. Let Ψ denote the family of all real functions $\Psi : R_+^3 \to R_+$ satisfying the following properties:
(Ψ_1) : Ψ is continuous on R_{+}^3, (Ψ_2) : $\Psi(1, 1, 1) = h > 1$,
(Ψ_3) : Let $\alpha, \beta \in R_+$ such that

$$\Psi - 3 - A : \alpha \geq \Psi(\beta, \beta, \alpha) = h\beta$$
$$\Psi - 3 - B : \alpha \geq \Psi(\beta, \alpha, \beta) = h\beta$$
$$\Psi - 3 - C : \forall \alpha > 0, \Psi(\alpha, 0, 0) > \alpha.$$

Let (X, τ) be a Hausdorff space and M a subset of X. In [7] is proved the following theorem which generalize results of Popa from [4],[5].

Theorem 1. *Let A,B,S,T be self-maps of M such that:*
(1.1) each of the pairs A,S and B,T are semi-compatible mappings,
(1.2) $A(M) \subset T(M)$ and $B(M) \subseteq S(M)$,
(1.3) $S(M)$ is d-complete,
(1.4) $d(Sx, Ty) > \Psi(d(Ax, By), d(Ax, Sx), d(By, Ty))$ for every $x, y \in M$ and $\psi \in \Psi$.
Then A,B,S,T have a unique common fixed point in M.
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The purpose of this paper is to prove a fixed point theorem which generalise Theorems for mappings satisfying an implicit relation.

2. Implicit relations.

Let $\mathcal{F}_4$ be the set of all real continuous functions $F : R_+^4 \to R$ satisfying the following conditions:
F_h : there exists $h \geq 1$ such that for every $u \geq 0, v \geq 0$ with $F(u, v, u, v) \geq 0$ or $F(u, v, v, u) \geq 0$ we have $u \geq hv$.
$F_u : F(u, u, 0, 0) < 0, \forall u > 0$.
Ex. 1. $F(t_1, ..., t_4) = t_1^2 - (at_2^2 + bt_3^2) - t_4^3$ where $a > 1, 0 \leq b < 1$.
F_h : Let $u \geq 0, v \geq 0$ and $F(u, v, u, v) = u^2 - v^3 - (av^2 + bu^2) \geq 0$ which implies $u^2 - (av^2 + bu^2) \geq 0$. Thus $u \geq h_1 v$ where $h_1 = \sqrt{(\frac{a}{1-b})} > 1$. Similarly, $F(u, v, v, u) \geq 0$ implies $u \geq h_2 v$, where $h_2 = \sqrt{a + b} > 1$.
For $h = min\{h_1, h_2\} > 1$ it follows that $u \geq hv$
$F_u : F(u, u, 0, 0) = u^2(1 - a) < 0 \quad \forall u > 0$.
Ex.2. $F(t) = t_1^k - (at_2^k + bt_3^k + ct_4^k) - (t_1 t_3)^{\frac{k}{2}} - (t_1 t_4)^{\frac{k}{2}}$ here $a > 1, 0 \leq b, c < 1$.
F_h: Let $u \geq 0, v > 0$ and $F(u, v, u, v) = u^k - [av^k + bu^k + cv^k] - u^k - (uv)^{\frac{k}{2}} \geq 0$ which implies $u^k - (av^k + bu^k + cv^k) \geq 0$ and thus $u \geq h_1 v$
where $h_1 = (\frac{a+c}{1-b})^{\frac{1}{k}} > 1$. Similarly, $F(u, v, v, u) \geq 0$ implies $u \geq h_2 v$ where $h_2 = (\frac{a+b}{1-c})^{\frac{1}{k}} > 1$. For $h = min\{h_1, h_2\}$ we have $u \geq hv$.
$F_u : F(u, u, 0, 0) = u^k(1 - a) < 0, \forall u > 0$.

3. Main results

Theorem 2. *Let A,B,S,T be self maps of M such that:*
-each of the pairs $\{A, S\}$ and $\{B, T\}$ are semi-compatible and
(3.1) $\qquad F(d(Sx, Ty), d(Ax, By), d(Ax, Sx), d(By, Ty)) \geq 0$,
for all x,y in M where $F \in \mathcal{F}_4$.

If there exists u,v and z in M such that $Au = Su = Bv = Tv = z$, then $Az = Bz = Sz = Tz = z$.

Proof. Since A and S are semi-compatible mappings and $Au = Su = z$, by property (D_1) , we have $Az = ASu = SAu = Sz$. From (3.1) we have successively

$$F(d(Sz, Tv), d(Az, Bv), d(Az, Sz), d(Bv, Tv)) \geq 0$$
$$F(d(Sz,z), d(Sz,z), 0, 0) \geq 0$$

a contradiction of (Fu) if $d(Sz, z) \neq 0$. Thus Sz=z. Similarly, $Tz = z$. Thus $Sz = Az = z$ and $Tz = Bz = z$.

Theorem 3. *Let A,B,S,T be self maps of M such that (3.1) holds for each $x, y \in M$, where F satisfies condition (F_u). Then A,B,S,T have at most one common fixed point.*

Proof. Suppose that A,B,S,T have two common fixed points z,z' with $z \neq z$.
Then by (3.1) we have sucessively

$$F(d(Sz, Tz'), d(Az, Bz'), d(Az, Sz), d(Bz', Tz')) \geq 0$$
$$F(d(z, z'), d(z, z'), 0, 0) \geq 0$$

a contradiction of (F_u) if $d(z, z') \neq 0$. Then z=z'.

Theorem 4. *Let A,B,S,T be self-maps of M satisfying the conditions (1.1); (1.2);(1.3) and (3.1). Then A,B,S,T have a unique common fixed point.*

Proof. For an arbitrary point x_0 in M, by (1.2) we define a sequence $\{x_n\}$ in M such that for n=0,1,2,...

$$(3.2)\begin{cases} Tx_{2n+1} = Ax_{2n} = y_{2n} \\ Sx_{2n+2} = Bx_{2n+1} = y_{2n+1}. \end{cases}$$

Define $d_n = d(y_n, y_{n+1})$ for all n=0,1,2,...Then by (3.2) we have successively

$$F(d(Sx_{2n+2}, Tx_{2n+1}), d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1})) \geq 0$$
$$F(d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n})) \geq 0$$
$$F(d_{2n}, d_{2n+1}, d_{2n+1}, d_{2n}) \geq 0$$

By (F_h) it follows that $d_{2n} > hd_{2n+1}$ which implies $d_{2n+1} \leq \frac{1}{h} d_{2n}$.
Similarly, by (3.1) and (Fh) it follows that
$d_{2n+2} \leq \frac{1}{h} d_{2n+1}$.
By induction gives
$d_n \leq (\frac{1}{h})^n d_0$ for all $n \in N$.
Since $h > 1$ this implies that $\sum_{n=1}^{\infty} d_n$ is convergent. Thus $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent. Since, in addition S(M) is d-complete, the sequence $\{y_n\}$ converges to some z in S(M), hence the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\}$, of $\{y_n\}$, also converge to z. Let Su=z for same $u \in M$. Putting x=u and $y = x_{2n+1}$ in (3.1) we have:
(3.3) $F(d(Su, Tx_{2n+1}), d(Au, Bx_{2n+1}), d(Au, Su)d(Bx_{2n+1}, Tx_{2n+1})) \geq 0$.
Letting $n \to \infty$ in (3.3), we get

$F(0, d(Au, z), d(Au, z), 0) \geq 0$

which implies by (F_b) that $Au = z$. Since $z \in A(M) \subset T(M)$, there exists a point v in M such that Au=Tv. Again, replacing x by u and y by v in (3.1), we have successively

$$F(d(Su, Tv), d(Au, Bv), d(Au, Su), d(Bv, Tv)) \geq 0$$
$$F(0, d(z, Bv), 0, d(Bv, z)) \geq 0$$

which implies by (F_b) that Bv=z. Therefore, we have Au=Su=Bv=Tv and by Theorem 2 it follows that z is a common fixed point of A,B,S and T.
The uniqueness of the common fixed point follows from Theorem 3.

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