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Dedicated to Costică MUSTĂŢA on his 60^{th} anniversary

SCHWARZSCHILD'S METRIC GENERATED BY A BODY WITH MASS m_1 PERTURBED BY A BODY WITH MASS m_2 FROM A FIXED DISTANCE

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Abstract. This paper approaches the modification a relativity metrics of spherical symetry generated by a body of mass m_1 due to a for off spherical deals with the translation of the reference frame $O'x'y'z'$with the origin in m_2 is done in the reference frame $Oxyz$ with the origin in m_1 .

The stringht line OO' being the support of the axes $O'x'$ and Ox. In the present case, distance $OO' = a$ is considered constant in time.

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The translation of the axes of coordinates

Let us consider two spherical bodies of mass m_1 and m_2 situated at a fixed distance a much bigger than their radii.

In this case a reference frame can be chosen for both bodies, so that mass m_1 should be located in the origin O of the system of axes $Oxyz$, and mass m_2 should be in the origin O' of the system of axes $O'x'y'z'$.

A spherical body of mass m_1 situated in the origin of the reference frame $Oxyzt$ determines a Schwarzschild metrics in its exterior [1].

$$ds_1^2 = g_{ij}dx^i dx^j = \left(1 - \frac{2m_1}{r}\right) dt^2 - \left(1 - \frac{2m_1}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \quad (1)$$

where

$$x^0 = t = ct_{ph}; \quad x^1 = r = r_{ph}; \quad x^2 = \theta = \theta_{ph}; \quad x^3 = \phi = \phi_{ph} \quad (2)$$

are the geometrized generalized spherical coordinates. The geometrized time and mass have length dimensions (in meters).

$$m = \frac{G\, M_{ph}}{c^2} \tag{3}$$

where index ph shows us that the respective magnitude is measured in physical units while c and Grepresents the speed of the light and the universal constant of gravity, respectively.

The body of mass m_2 with the mass center situated in the origin of the reference point $O't'x'y'z't'$ determiness the metrics

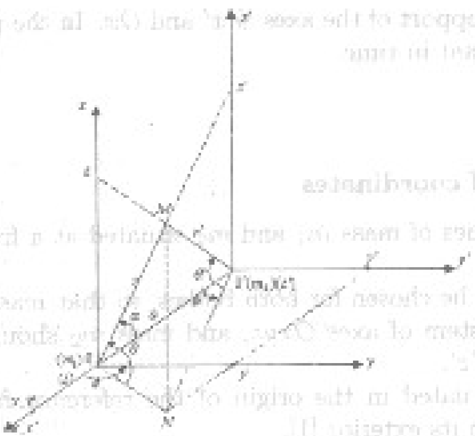
$$ds_2^2 = \left(1 - \frac{2m_2}{r'}\right) dt'^2 - \left(1 - \frac{2m_2}{r'}\right)^{-1} dr'^2 - r'^2 d\theta'^2 - r'^2 \sin^2\theta' \, d\phi'^2 \tag{4}$$

The length unit in both cases in the meter measured by a for all observer.

A point M from the space-time Universe has the Cartesian coordinates (t, x, y, z) measured from O and (t', x', y', z') from O'[2]. We have noted the radial distances by

$$OM = r; \quad O'M = r'; \quad OO' = a \tag{5}$$

(see the next figure)



Between the spherical coordinates and the Cartesian coordinates of the two reference point there is the relationschip:

$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\cos\theta sin\phi \\ z = r\sin\theta \end{cases} \qquad \begin{cases} x' = r'\cos\theta'\cos\phi' \\ y' = r'\cos\theta' sin\phi' \\ z' = r'\sin\theta' \end{cases} \tag{6}$$

It is to be notice that:

$$\begin{cases} x' = a + x \\ y' = y \\ z' = z \end{cases} \tag{7}$$

By applying Pytagoras' generalized theorem under the trigonometrical form in the triangle MOO', we have

$$r'^2 = a^2 + r^2 - 2ar\cos\alpha \tag{8}$$

is results

$$r' = E = \sqrt{a^2 + r^2 + 2ar\cos\theta\cos\phi} \tag{9}$$

An event occurring in the point M at the moment $t_{M,x}$ measured after the clock in M, will be seen in O after the arrival of the luminous signal coming with the speed of light c at the moment $t_{ph} + \frac{r}{c}$, and in O' at the moment $t'_{ph} + \frac{r'}{c}$. In geometrized units

$$t_M = t + r = t' + r' \tag{10}$$

of which

$$t' = t + r - E \tag{11}$$

As $z = z'$ from (6) and (9) results

$$\sin\theta' = \frac{r\sin\theta}{E} \quad \text{and} \quad \cos\theta' = \frac{F}{E} \tag{12}$$

where

$$F = \sqrt{a^2 + r^2\cos^2\theta + 2ar\cos\theta\cos\phi} \tag{13}$$

In the same way, from $y' = y$ and (6) and (13) we have

$$\sin\phi' = \frac{r\cos\theta\sin\phi}{F} and \cos\phi' = \frac{a + r\cos\theta\cos\phi}{F} \tag{14}$$

The passing of Schwarzschild's metrics associated with mass m_2 to the coordinates associated with the reference point with the origin in m_1

Let us suppose, for the simplification of the calculations, that $a = OO'$ =constant in time. Therefore

$$\frac{da}{dt} = 0 \tag{15}$$

Diferentiating the linking relations (9), (11), (12), (13), (14) and replacing the magnitudes by prime ('), in metrics (4) we get the metric ds_2^2.

The results obtained in this way we unified in the following theorem.

Theorem. *The Schwarzschild's metrics generated by a body with mass m_1 is perturbed by a body with mass m_2 situated at a fixed distance a, with the supplementary metric:*

$$\begin{aligned} ds_2^2 = h_{ij}dx^i dx^j = h_{00}dt^2 + h_{11}dr^2 + h_{22}d\theta^2 + h_{33}d\phi^2 + \\ +2h_{01}dtdr + 2h_{02}dtd\phi + 2h_{03}dtd\phi + \\ +2h_{12}drd\theta + 2h_{13}drd\phi + 2h_{23}d\theta d\phi \end{aligned} \tag{16}$$

where the metrics ds_2^2 are expressed in the coordinates corresponding to the reference system O_{txyz}, where h_{ij} are expressed as follows:

$$h_{00} = 1 - \frac{2m_2}{E} \tag{17}$$

$$\begin{aligned} h_{11} \;=\; & \frac{1}{E^2}\left[\left(1-\frac{2m_2}{E}\right)(E - r - a\cos\theta\cos\phi) - \left(1-\frac{2m_2}{E}\right)^{-1}(r + a\cos\theta\cos\phi)^2\right] - \\ & -\frac{a^2}{F^2}\sin^2\theta\left[\frac{(a + r\cos\theta\cos\phi)^2}{E^2} + \frac{r^2\cos^2\theta\sin^2\phi}{F^2}\right] \end{aligned} \tag{18}$$

$$\begin{aligned} h_{22} = & \frac{1}{E^2}a^2r^2\sin^2\theta\cos^2\phi\left[\left(1-\frac{2m_2}{E}\right) - \left(1-\frac{2m_2}{E}\right)^{-1}\right] - \\ & -\frac{r^2}{F^2}\left\{\frac{1}{E^2}[(a^2+r^2)\cos\theta + ar\cos\phi(1-\cos^2\theta)]^2 + \right. \\ & \left. + \frac{r^2\sin^4\theta\sin^2\phi}{F^2(a + r\cos\theta\cos\phi)^2}[a^2 + r^2\cos^2\theta + r(a-r)\cos\theta\cos\phi]^2\right\} \end{aligned} \tag{19}$$

$$h_{33} = \frac{1}{E^2} a^2 r^2 \cos^2\theta \sin^2\phi \left[\left(1 - \frac{2m_2}{E}\right) - \left(1 - \frac{2m_2}{E}\right)^{-1} \right] -$$

$$-\frac{r^4}{F^2} \sin^2\theta \cos^2\theta \left\{ \frac{a^2 \sin^2\phi}{E^2} + \frac{1}{F^2 (a + r\cos\theta\cos\phi)^2} \cdot \right.$$

$$\left. \cdot \left[ar\cos\theta(1 + \cos^2\phi) + (a^2 + r^2\cos^2\theta)\cos\phi \right]^2 \right\} \tag{20}$$

$$h_{01} = \left(1 - \frac{r + a\cos\theta\cos\phi}{E}\right)\left(1 - \frac{2m_2}{E}\right) \tag{21}$$

$$h_{02} = \frac{1}{E} ar \sin\theta\cos\phi \left(1 - \frac{2m_2}{E}\right) \tag{22}$$

$$h_{03} = \frac{1}{E} ar \cos\theta \sin\phi \left(1 - \frac{2m_2}{E}\right) \tag{23}$$

$$h_{12} = \frac{1}{E^2} ar \sin\theta\cos\phi \cdot$$
$$\cdot \left[\left(1 - \tfrac{2m_2}{E}\right)(E - r - a\cos\theta\cos\phi) + \left(1 - \tfrac{2m_2}{E}\right)^{-1} (r + a\cos\theta\cos\phi) \right] -$$

$$-\frac{ar}{F^2} \sin\theta (a + r\cos\theta\cos\phi) \left\{ \frac{1}{E^2} \left[(a^2 + r^2)\cos\theta + ar\cos\phi(1 + \cos^2\theta) \right] - \right.$$

$$\left. - \frac{r^2 \sin^2\theta\cos\theta\sin^2\phi}{F^2 (a + r\cos\theta\cos\phi)^2} \left[a^2 + r^2\cos^2\theta + r(a - r)\cos\theta\cos\phi \right] \right\} \tag{24}$$

$$h_{13} = \frac{1}{E^2} ar \cos\theta\sin\phi \cdot$$
$$\cdot \left[\left(1 - \tfrac{2m_2}{E}\right)(E - r - a\cos\theta\cos\phi) + \left(1 - \tfrac{2m_2}{E}\right)^{-1} (r + a\cos\theta\cos\phi) \right] -$$

$$-\frac{ar^2}{F^2} \sin^2\theta\cos\theta\sin\phi \left\{ \frac{a(a + r\cos\theta\cos\phi)}{E^2} + \frac{r\cos\theta}{F(a + r\cos\theta\cos\phi)} \cdot \right.$$

$$\left. \cdot \left[ar\cos\theta(1 + \cos^2\phi) + (a^2 + r^2\cos^2\theta)\cos\phi \right] \right\} \tag{25}$$

$$h_{23} = \frac{1}{E^2} a^2 r^2 \sin\theta \cos\theta \sin\phi \cos\phi \left[\left(1 - \frac{2m_2}{E}\right) - \left(1 - \frac{2m_2}{E}\right)^{-1} \right] -$$

$$- \frac{r^3}{F^2} \sin\theta \cos\theta \sin\phi \left\{ \frac{a}{E^2} [(a^2 + r^2)\cos\theta + ar\cos\phi(1 + \cos^2\theta)] - \right.$$

$$- \frac{ar^3 \sin\theta \cos\theta \sin\phi}{F^2} \left\{ \frac{1}{E^2} [(a^2 + r^2)\cos\theta + ar(1 + \cos^2\theta)\cos\phi] - \right.$$

$$\left. - \frac{r \sin^2\theta}{F^2 (a + r\cos\theta\cos\phi)} [(a^2 + r^2\cos^2\theta)\cos\phi + ar\cos\theta(1 + \cos^2\phi)] \right\} \qquad (26)$$

□

REFERENCES

[1]. Landau L., Lifchitz E., "*Physique théoretique. Théorie du champ*". Ed. Mir, Moscou, 1966

[2] Sass I. H. A., " *Late stages in Stellar Evolution*", Monographical Booklets in Applied&Computer Mathematics, PAMM, Budapest, 2002

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