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THE CONVERGENCE OF MANN ITERATION FOR AN ASYMPTOTIC HEMICONTRACTIVE MAP

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Abstract. We prove that the Mann iteration converges to a fixed point of an asymptotic hemicontractive map.

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1. Introduction.

Let X be a real Hilbert space, let $B \subset X$ be a nonempty, convex set. Let $T : B \mapsto B$ be a map. Let $\mathbb{N} = \{1, 2, ...\}$. Let $x_1 \in B$, be an arbitrary fixed point. We consider the iteration

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \forall n \in \mathbb{N}. \tag{1}$$

The sequence $(\alpha_n)_{n\geq 1}$ satisfies:

$$(\alpha_n)_{n\geq 1} \subset (0,1), \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0.$$

A prototype for $(\alpha_n)_{n\geq 1}$ is $(1/n)_{n\geq 1}$. Iteration (1) is known as Mann iteration, see [8]. We consider the following iteration, known as Ishikawa iteration, see [6]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \tag{Ish}$$

$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \ n \in \mathbb{N},$$

where $(\alpha_n)_n, (\beta_n)_n \subset (0, 1)$. Choosing $\beta_n = 0$, $\forall n \in \mathbb{N}$, from (Ish) we get (1).

Let us denote by $F(T)$ the set of fixed points of the operator T. We need the following definition, see for example [10]:

Definition 1. *The map $T : B \to B$ is called asymptotically hemicontractive with sequence $(k_n)_n$, if and only if $\lim_{n\to\infty} k_n = 1$, $F(T) \neq \emptyset$ such that*

$$\|T^n x - x^*\|^2 \leq k_n \|x - x^*\|^2 + \|x - T^n x\|^2, \forall x \in B, x^* \in F(T), \forall n \in \mathbb{N}. \qquad (2)$$

In this note, we will consider

$$k_n < 1, \forall n \in \mathbb{N}.$$

In context of Hilbert spaces, the convergence of Ishikawa iteration (Ish) to the fixed point of T, when we deal with a asymptotically hemicontractive map (with $k_n > 1, \forall n \in N$), could be found in [10]. A convergence result in normed spaces for (Ish) (with $(\alpha_n)_n, (\beta_n)_n \subset (0,1)$ being not convergent to zero), could be found in [5]. Anyway, in [5], we deal with an asymptotic hemicontractive like map, which is not the same as in Definition 2 with condition (2). Let us remark that:

(i) In [10] and [5], the sequences $(\alpha_n)_n, (\beta_n)_n \subset (0,1)$ are not convergent to zero, because of the existence of an $\varepsilon > 0$ such that $\varepsilon \leq \alpha_n \leq \beta_n, \forall n \in \mathbb{N}$. From the convergence of (Ish) (which is proved in [10] for Hilbert spaces and T asymptotic hemicontractive), with that sequences $(\alpha_n)_n, (\beta_n)_n$, we can not deduces convergence of Mann iteration. It is impossible to have $\beta_n = 0, \forall n \in \mathbb{N}$.

(ii) According to [5], convergence results of (Ish) for asymptotic hemicontractive maps, exist only in [10], (and of course in [5]). So far as we know, no other papers are dealing with asymptotic hemicontractive maps and Mann iteration (1).

This two reasons lead us to remark a lack of a convergence result which deal with Mann iteration (1) for an asymptotic hemicontractive map. In particular for a map asymptotically hemicontractive with sequence $(k_n)_n, k_n < 1, \forall n \in \mathbb{N}$. Our aim is to prove a result which deal with the convergence of Mann iteration (1) for an asymptotic hemicontractive map as in Definition 1.

The following lemma can be found in [11] as Lemma 4. Also, it can be found in [12] as Lemma 1.2, with an other proof. In [1] can be found as Lemma 2.

Lemma 1 [1], , [11], [12] *Let $(a_n)_{n\geq 1}$ be a nonnegative sequence which verifies*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \sigma_n,$$

where $(\lambda_n)_{n\geq 1} \subset (0,1)$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

The following result is from [6].

Lemma 2 [6]. *Let X be a real Hilbert space, the following relation is true for all $x, y \in X$, and for all $\lambda \in (0,1)$:*

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \qquad (3)$$

2. Main result

We are able now to give the following result:

Theorem 1. *Let X be a real Hilbert space and let $B \subset X$ be a nonempty convex bounded set, and $T : B \to B$ be an asymptotic hemicontractive map with $(\alpha_n)_{n\geq 1}$ $k_n < 1$. Let $\{x^*\} = F(T)$. If $(\alpha_n)_n$ given by (1) verifies $\alpha_n = o((1-k_n))$, then the Mann iteration $(x_n)_{n\geq 1}$, given by (1), is convergent to x^*, for all $x_1 \in B$.*

Proof. The sequence $(\|x_n - x^*\|)_n$ is well-defined, because x^* is unique. Using (3) and (2) we have, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n x_n - T^n x^*)\|^2 \\
&= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|T^n x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T^n x_n - x_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(k_n\|x_n - x^*\|^2 + \|T^n x_n - x_n\|^2) \\
&\quad -\alpha_n(1-\alpha_n)\|T^n x_n - x_n\|^2 \\
&= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n\|x_n - x^*\|^2 + \alpha_n\|T^n x_n - x_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|T^n x_n - x_n\|^2 \\
&= (1-\alpha_n(1-k_n))\|x_n - x^*\|^2 + \alpha_n^2\|T^n x_n - x_n\|^2 \\
&\leq (1-\alpha_n(1-k_n))\|x_n - x^*\|^2 + \alpha_n^2 M.
\end{aligned}$$

The last inequality is true, since the sequence $(\|T^n x_n - x_n\|^2)_{n\geq 1}$ is bounded, because B is bounded. There exists $M > 0$ such that $\|T^n x_n - x_n\|^2 < M$, for all $n \in \mathbb{N}$. We denote by $a_n := \|x_n - x^*\|$, $\lambda_n = \alpha_n(1-k_n)$, $\sigma_n = \alpha_n^2 M$, $\forall n \in \mathbb{N}$. Because $\alpha_n = o((1-k_n))$, we have $\lim_{n\to\infty} \frac{\alpha_n}{1-k_n} M = 0$. All the conditions from Lemma 2 are fulfilled. Thus $\lim_{n\to\infty} a_n = 0$. The proof is complete.□

In Theorem 4 we don't need any Lipschitz condition for T as in [10]

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