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Dedicated to Costică MUSTĂŢA on his 60th anniversary

APPLICATION OF FIBER PICARD OPERATORS TO INTEGRAL EQUATIONS

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Abstract. We use the existence and uniqueness fixed point theorems for operators of type $f : X^n \to X$ to study the existence and uniqueness of the solution for some integral equations. Also, we will use the fiber Picard operator technique to obtain derivability results with respect to some parameter.

MSC: 47H10

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1.Introduction

In this paper we study the integral equation of quadratic type

$$x(t) = 1 + \lambda \int_t^1 x(s)\, x(s-t)\, ds, \qquad t \in [0;1]. \tag{1}$$

This equation was presented by M.S. Wertheim, a physicist at the Los Alamos Scientific Laboratory, as a simplified model of certain equations arising from Statistical Mechanics and may be considered as a special case of the Percus-Yevick equation discussed by M.S. Wertheim in [6].

The purpose of this paper is to use the fixed point theorems for operators on cartesian product of metric spaces to prove the existence and uniqueness of solution for equation (1) in some special invariant sets constructed from the properties of the solution. Using the fiber Picard operator technique we obtain results concerning the continuity and derivability of solution respect to the parameter λ. The advantage of using the fixed point theorems for operators on cartesian product is the fact that it will be obtained more iterative methods for approximating solution than in the case of using contraction principle.

In the last part of the paper we give some generalization of the results obtained in the first part for the general integral equation

$$x(t) = g(t) + \lambda \int_t^1 K(t,s)\, x(s)\, x(s-t)\, ds, \qquad t \in [0;1]. \tag{2}$$

2. Fixed point theorems

Let X be a nonempty set and $A : X \to X$ an operator. We note by:

$P(X) := \{Y \subset X \mid \quad Y \neq \emptyset\}$

$F_A := \{x \in X \mid \quad A(x) = x\}$ - the fixed point set of A.

For $\overline{x} = (x_0, x_1, \ldots, x_{k-1}) \in X^k$ and an operator $f : X^k \to X$, we can construct the following sequences:

$$\begin{array}{ll} y_0 & = f(x_0, x_1, \ldots, x_{k-1}), \\ y_1 & = f(y_0, y_0, \ldots, y_0), \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n+1} & = f(y_n, y_n, \ldots, y_n), \end{array} \tag{3}$$

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \tag{4}$$

and the following operators:

$$\begin{array}{c} \tilde{f} : X \to X \\ \tilde{f}(x) = f(x, \ldots, x) \end{array} \tag{5}$$

$$\begin{array}{c} A_f : X^k \to X^k \\ (u_1, \ldots, u_k) \longmapsto (u_2, \ldots, u_k, f(u_1, \ldots, u_k)). \end{array} \tag{6}$$

For the sequences (4) and (3) we have

$$y_{n+1} = \tilde{f}^n(y_0)$$

$$(x_{n+1}, \ldots, x_{n+k}) = A_f^n(x_0, x_1, \ldots, x_{k-1}).$$

Definition 2.1 (I.A. Rus [4]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) Picard operator if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$,

(b) the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges (uniformly) to x^*, for all $x \in X$.

Definition 2.2 (I.A. Rus [4]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) weakly Picard operator if:

(a) the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges (uniformly), for all $x \in X$,

(b) the limit (which may depend on x) is a fixed point of A.

Theorem 2.1. (M.A. Şerban [5]) Let (X, d) be a complete metric space and $f : X^k \to X$. Suppose there exist $q_i \in \mathbb{R}_+$, $i = \overline{1,k}$, with $\alpha = \sum\limits_{i=1}^{k} q_i < 1$ such that

$$d\left(f\left(\overline{x}\right), f\left(\overline{y}\right)\right) \leq \sum_{i=1}^{k} q_i d\left(x_i, y_i\right)$$

for any $\overline{x} = (x_1, \ldots, x_k)$, $\overline{y} = (y_1, \ldots, y_k) \in X^k$. Then:

(a) the operator $\tilde{f} : X \to X$, defined by (5), is a Picard operator, i.e. $F_f = \{x^*\}$;

(b) the sequence $(y_n)_{n\in\mathbb{N}}$, defined by (3), converges to x^* and we have the estimation

$$d\left(y_n, x^*\right) \leq \frac{\alpha^{n+1}}{1-\alpha} \cdot \max_{i=\overline{1,k}} \left\{d\left(x_i, f\left(\overline{x}\right)\right)\right\}, \tag{7}$$

for any $\overline{x} = (x_1, \ldots, x_k) \in X^k$;

(c) the operator $A_f : X^k \to X^k$, defined by (6), is a Picard operator, $F_{A_f} = \{(x^*, \ldots, x^*)\}$;

(d) the sequence $(x_n)_{n\in\mathbb{N}}$, defined by (4), converges to x^* and we have the estimation

$$d\left(x_n, x^*\right) \leq k \cdot d_0 \cdot \frac{\alpha^{\left[\frac{n}{k}\right]}}{1-\alpha}, \tag{8}$$

for any $\overline{x} = (x_0, \ldots, x_{k-1}) \in X^k$, where $d_0 = \max\limits_{i=\overline{1,k}} \left\{d\left(x_{i-1}, x_i\right)\right\}$.

Theorem 2.2 [Fiber φ-contraction theorem] (M.A. Şerban [5]) Let (X_j, d_j), $j = \overline{0,p}$, $p \geq 1$, be some metric spaces. Let

$$A_j : X_0 \times \ldots \times X_j \to X_j, \qquad j = \overline{0,p},$$

be some operators such that:

(i) the spaces (X_j, d_j), $j = \overline{1,p}$, are complete metric spaces;

(ii) the operator A_0 is (weakly) Picard operator;

(iii) there exist $\varphi_j : \mathbb{R}_+ \to \mathbb{R}_+$ subadditive (c)-comparison functions such that the operators $A_j(x_0, ..., x_{j-1}, \cdot)$ are φ_j-contractions, $j = \overline{1,p}$;

(iv) the operators A_j are continuous with respect to $(x_0, ..., x_{j-1})$ for all $x_j \in X_j$, $j = \overline{1,p}$.

Then the operator $B_p = (A_0, ..., A_p)$ is (weakly) Picard operator. Moreover if A_0 is a Picard operator and

$$F_{A_0} = \{x_0^*\}, \qquad F_{A_1(x_0^*, \cdot)} = \{x_1^*\}, ..., F_{A_p(x_0^*, ..., x_{p-1}^*, \cdot)} = \{x_p^*\}$$

then

$$F_{B_p} = \{(x_1^*, x_2^*, ..., x_p^*).$$

3. Application to integral equation

We consider the Banach space $X = \left(C\left([0;1] \times [0;\lambda_0], \mathbb{R}\right), \|\cdot\|_C\right)$ of continuous functions with the uniform norm, where $\lambda_0 > 0$, and we define the operator

$$A : X \times X \to X$$
$$A(x,y)(t,\lambda) = 1 + \frac{\lambda}{2}\left[\int_t^1 x(s,\lambda)\, y(s-t,\lambda)\, ds + \int_t^1 x(s-t,\lambda)\, y(s,\lambda)\, ds\right] \tag{9}$$

Proposition 3.1. Let $x \in C[0;1]$ a solution of equation (1). Then

$$I(x) = \int_0^1 x(t)\, dt$$

satisfies the quadratic equation

$$\lambda I^2(x) - 2I(x) + 2 = 0. \tag{10}$$

Proof. We integrate the equation (1):

$$\begin{aligned} I(x) &= \int_0^1 x(t)\, dt && = 1 + \lambda \int_0^1 \int_t^1 x(s)\, x(s-t)\, ds dt \\ &= 1 + \lambda \int_0^1 ds \int_0^s x(s)\, x(s-t)\, dt && = 1 + \lambda \int_0^1 x(s) \left(\int_0^s x(z)\, dz\right) ds \\ &= 1 + \frac{\lambda}{2} \cdot \left(\int_0^1 x(s)\, ds\right)^2 && = 1 + \frac{\lambda}{2} \cdot I^2(x) \end{aligned}$$

and we obtain the conclusion.□

Remark 3.1.

(a) There are no solutions if $\lambda > \frac{1}{2}$;

(b) If $\lambda = 0$, we have the trivial solution $x(t) \equiv 1$;

(c) If $x, y \in C\left([0;1] \times [0;\lambda_0], \mathbb{R}_+\right)$ then $A(x,y) \geq 1$.

Since $I(x)$ satisfies the quadratic equation (10) then

$$I(x) = I_{1,2} = \frac{1 - \sqrt{1 - 2\lambda}}{\lambda}$$

and this lead us to consider the following sets:

$$\begin{aligned} Y_i = \{x \in X : x(t,\lambda) \geq 1, \quad \forall (t,\lambda) \in [0;1] \times [0;\lambda_0], \\ I(x) = I_i,\ for\ \lambda \neq 0,\ \ x(t,0) = 1,\ \forall t \in [0;1]\} \end{aligned} \tag{11}$$

for $i = 1, 2$.

Proposition 3.2 The sets Y_i, $i = 1, 2$, defined by (11), are invariant sets for operator A, defined by (9).

Proof. Let $x, y \in Y_i$. It is obvious that $A(x,y)(t,\lambda) \geq 1$, $\forall (t,\lambda) \in [0;1] \times [0;\lambda_0]$ and $A(x,y)(t,0) = 1$, $\forall t \in [0;1]$. For $\lambda \neq 0$, we have:

$$\begin{aligned}
\int_0^1 A(x,y)(t,\lambda)\,dt &= 1 + \tfrac{\lambda}{2} \cdot \left[\int_0^1 \int_t^1 x(s,\lambda)\, y(s-t,\lambda)\, dsdt + \int_0^1 \int_t^1 y(s,\lambda)\, x(s-t,\lambda)\, dsdt\right] = \\
&= 1 + \tfrac{\lambda}{2} \cdot \left[\int_0^1 ds \int_0^s x(s,\lambda)\, y(s-t,\lambda)\, dt + \int_0^1 ds \int_0^s y(s,\lambda)\, x(s-t,\lambda)\, dt+\right] = \\
&= 1 + \tfrac{\lambda}{2} \cdot \left[\int_0^1 x(s,\lambda) \left(\int_0^s x(z,\lambda)\, dz\right) ds + \int_0^1 y(s,\lambda) \left(\int_0^s y(z,\lambda)\, dz\right) ds\right] = \\
&= 1 + \tfrac{\lambda}{2} \cdot I(x) \cdot I(y) = \\
&= 1 + \tfrac{\lambda}{2} \cdot I_i^2 = I_i
\end{aligned}$$

which proves the conclusion.□

Theorem 3.1. If $0 < \lambda_0 < \frac{3}{8}$ then:

(a) There is a unique $x^* \in C([0;1], \mathbb{R}_+)$ solution of equation (1) such that

$$\int_0^1 x^*(t)dt = I_1$$

for $\lambda \in [0;\lambda_0]$ fixed;

(b) for any $x_0, x_1 \in Y_1$ the sequence $(y_n)_{n\in\mathbb{N}}$

$$\begin{aligned}
y_0 &= A(x_0, x_1), \\
y_1 &= A(y_0, y_0), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_{n+1} &= A(y_n, y_n),
\end{aligned}$$

converges uniformly to x^* and we have the estimation

$$\|y_n - x^*\| \leq \frac{\alpha^{n+1}}{1-\alpha} \cdot \max\{\|x_0 - x_1\|, \|x_1 - A(x_0, x_1)\|\},$$

where $\alpha = 2\left(1 - \sqrt{1 - 2\lambda_0}\right)$;

(c) for any $x_0, x_1 \in Y_1$ the sequence $(x_n)_{n\in\mathbb{N}}$

$$x_{n+2} = A(x_n, x_{n+1})$$

converges uniformly to x^* and we have the estimation

$$\|x_n - x^*\| \leq \frac{\alpha^{\left[\frac{n}{2}\right]}}{1-\alpha} \cdot \max\{\|x_0 - x_1\|, \|x_1 - A(x_0, x_1)\|\},$$

for the same α;

(d) $x^* \in C^1\left([0;1]\times[0;\lambda_0]\right)$.

Proof. From Proposition 3.2 we have that $A : Y_1 \times Y_1 \to Y_1$. The set Y_1 is a closed set in a Banach space which means that is a complete metric space. For $\overline{x} = (x_1, x_2)$, $\overline{y} = (y_1, y_2) \in Y_1 \times Y_1$ we have

$$\begin{aligned}\|A(x_1,x_2) - A(y_1,y_2)\| &\leq \lambda \cdot I_1 \cdot [\|x_1 - y_1\| + \|x_2 - y_2\|] \leq \\ &\leq \tfrac{\alpha}{2} \cdot [\|x_1 - y_1\| + \|x_2 - y_2\|]\end{aligned}$$

which prove the conditions of Theorem 2.1 and therefore we obtain (a)-(c).

To prove (d) we will use the *Fiber φ-contraction Theorem.* We consider the sets $X_0 = Y_1$, $X_1 = X$ and operators:

$$A_0 : X_0 \to X_0$$
$$A_0(x)(t,\lambda) = 1 + \lambda \int_t^1 x(s,\lambda)\, x(s-t,\lambda)\, ds$$

which is a Picard operator since $A_0 = \tilde{A}$, defined as in (5), $F_{A_0} = \{x^*\}$. The second operator is obtained by formally derivating of operator A_0 with respect to λ

$$A_1 : X_0 \times X_1 \to X_1$$
$$A_1(x,y)(t,\lambda) = \int_t^1 x(s,\lambda)\, x(s-t,\lambda)\, ds + \lambda \left[\int_t^1 x(s,\lambda)\, y(s-t,\lambda)\, ds + \int_t^1 y(s,\lambda)\, x(s-t,\lambda)\, ds\right].$$

For any $x \in X_0$ and $y_1, y_2 \in X_1$ we have

$$\begin{aligned}\|A_1(x,y_1) - A_1(x,y_2)\| &\leq 2\lambda \cdot I_1 \cdot \|y_1 - y_2\| \leq \\ &\leq \alpha \cdot \|y_1 - y_2\|\end{aligned}$$

which shows that $A_1(x,\cdot)$ is a φ-contraction with $\varphi(t) = \alpha t$. From *Fiber φ-contraction Theorem* we deduce that the operator

$$B_1 : X_0 \times X_1 \to X_0 \times X_1$$
$$B_1 = (A_0, A_1)$$

is a Picard operator and $F_{B_1} = \{(x^*, y^*)\}$, where $\{y^*\} = F_{A_1(x^*,\cdot)}$. This means that the sequences

$$x_{n+1} = A_0(x_n)$$

$$y_{n+1} = A_1(x_n, y_n)$$

converge uniformly, $x_n \rightrightarrows x^*$, respectively $y_n \rightrightarrows y^*$, for any starting point $(x_0, y_0) \in X_0 \times X_1$. We remark if we take $y_0 = \frac{\partial x_0}{\partial \lambda}$ then $y_n = \frac{\partial x_n}{\partial \lambda}$ for all $n \in \mathbb{N}$. These imply that $y^* = \frac{\partial x^*}{\partial \lambda}$.

To prove that there exists $\frac{\partial x^*}{\partial t}$ and $\frac{\partial x^*}{\partial t} \in C\left([0;1] \times [0;\lambda_0]\right)$ we will use the same technique and in this case we consider the operator

$$A_1 : X_0 \times X_1 \to X_1$$
$$A_1\left(x,y\right)\left(t,\lambda\right) = -\lambda x\left(t,\lambda\right)x\left(0,\lambda\right) - \lambda\int\limits_t^1 x\left(s,\lambda\right)y\left(s-t,\lambda\right)ds.$$

The proof is complete.□

4. Generalization

In this section of the paper we will extend the results obtained in previous section to the integral equation

$$x\left(t\right) = g\left(t\right) + \lambda\int\limits_t^1 K\left(t,s\right)x\left(s\right)x\left(s-t\right)ds, \qquad t \in [0;1].$$

Suppose that the following assumptions hold:

(H1) $g \in C([0;1], \mathbb{R}_+)$;

(H2) $K \in C([0;1] \times [0;1], \mathbb{R})$;

(H3) $0 \leq K(t,s) \leq 1, \forall t, s \in [0;1]$.

Proposition 4.1 Let $x \in C\left[0;1\right]$ a solution of equation (2). Then

$$I\left(x\right) = \int\limits_0^1 x\left(t\right)dt$$

satisfies the inequation

$$\lambda I^2\left(x\right) - 2I\left(x\right) + 2G \geq 0, \tag{12}$$

where $G = \int\limits_0^1 g\left(t\right)dt$.

Proof. We integrate the equation (2):

$$\begin{aligned} I\left(x\right) &= \int\limits_0^1 x\left(t\right)dt && = \int\limits_0^1 g\left(t\right)dt + \lambda\int\limits_0^1\int\limits_t^1 K\left(t,s\right)x\left(s\right)x\left(s-t\right)dsdt \leq \\ &\leq G + \lambda\int\limits_0^1 ds\int\limits_0^s x\left(s\right)x\left(s-t\right)dt && = G + \lambda\int\limits_0^1 x\left(s\right)\left(\int\limits_0^s x\left(z\right)dz\right)ds = \\ &= G + \frac{\lambda}{2}\cdot\left(\int\limits_0^1 x\left(s\right)ds\right)^2 && = G + \frac{\lambda}{2}\cdot I^2\left(x\right) \end{aligned}$$

and we obtain the conclusion.□

Remark 4.1

(a) If $\lambda > \frac{1}{2G}$ then inequality (12) is always satisfied for any $I(x)$;

(b) If $\lambda = 0$, we have the trivial solution $x(t) \equiv g(t)$;

(c) If $x, y \in C([0;1] \times [0;\lambda_0], \mathbb{R}_+)$ then $A(x,y) \geq g(t)$.

Since $I(x)$ satisfies the inequation (12) then

$$I(x) \in (-\infty; I_1] \cup [I_2; +\infty)$$

where $I_{1,2} = \frac{1-\sqrt{1-2G\lambda}}{\lambda}$. We are interested to find the positive solution of integral equation (2), this lead us to consider the following set:

$$\begin{aligned} Y_1 = \{ & x \in X : x(t,\lambda) \geq m_g, \quad \forall (t,\lambda) \in [0;1] \times [0;\lambda_0], \\ & I(x) \leq I_1, \ for\ \lambda \neq 0, \ \ x(t,0) = g(t), \ \forall t \in [0;1] \} \end{aligned}$$

and the operator

$$A : X \times X \to X$$

$$A(x,y)(t,\lambda) = g(t) + \tfrac{\lambda}{2}\left[\int\limits_t^1 K(t,s)\, x(s,\lambda)\, y(s-t,\lambda)\, ds + \int\limits_t^1 K(t,s)\, x(s-t,\lambda)\, y(s,\lambda)\, ds \right].$$

Theorem 4.1 If $0 < \lambda_0 < \frac{3}{8G}$ then:

(a) There is a unique $x^* \in C([0;1], \mathbb{R}_+)$ solution of equation (2) such that

$$\int_0^1 x^*(t)dt \leq I_1$$

for $\lambda \in [0;\lambda_0]$ fixed;

(b) for any $x_0, x_1 \in Y_1$ the sequence $(y_n)_{n\in\mathbb{N}}$

$$\begin{aligned} y_0 &= A(x_0, x_1), \\ y_1 &= A(y_0, y_0), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n+1} &= A(y_n, y_n), \end{aligned}$$

converges uniformly to x^* and we have the estimation

$$\|y_n - x^*\| \leq \frac{\alpha^{n+1}}{1-\alpha} \cdot \max\{\|x_0 - x_1\|, \|x_1 - A(x_0,x_1)\|\},$$

where $\alpha = 2\left(1 - \sqrt{1 - 2\lambda_0 G}\right)$;

(c) for any $x_0, x_1 \in Y_1$ the sequence $(x_n)_{n\in\mathbb{N}}$

$$x_{n+2} = A(x_n, x_{n+1})$$

converges uniformly to x^* and we have the estimation

$$\|x_n - x^*\| \leq \frac{\alpha^{\left[\frac{n}{2}\right]}}{1-\alpha} \cdot \max\left\{\|x_0 - x_1\|, \|x_1 - A(x_0, x_1)\|\right\},$$

for the same α;

(d) If $g \in C^1([0;1], \mathbb{R}_+)$ and $\frac{\partial K}{\partial t} \in C([0;1], \mathbb{R})$ then $x^* \in C^1([0;1] \times [0;\lambda_0])$.

Proof. It is easy to check that $A(Y_1, Y_1) \subseteq Y_1$. The set Y_1 is a closed set in a Banach space which means that is a complete metric space. For $\overline{x} = (x_1, x_2)$, $\overline{y} = (y_1, y_2) \in Y_1 \times Y_1$ we have

$$\begin{aligned} \|A(x_1, x_2) - A(y_1, y_2)\| &\leq \lambda \cdot I_1 \cdot [\|x_1 - y_1\| + \|x_2 - y_2\|] \leq \\ &\leq \tfrac{\alpha}{2} \cdot [\|x_1 - y_1\| + \|x_2 - y_2\|] \end{aligned}$$

which prove the conditions of Theorem 2.1 and therefore we obtain (a)-(c).

To prove (d) we will use the *Fiber φ-contraction Theorem.* We consider the sets $X_0 = Y_1$, $X_1 = X$ and operators:

$$A_0 : X_0 \to X_0$$
$$A_0(x)(t,\lambda) = g(t) + \lambda \int_t^1 K(t,s)\, x(s,\lambda)\, x(s-t,\lambda)\, ds$$

which is a Picard operator since $A_0 = \tilde{A}$, defined as in (5), $F_{A_0} = \{x^*\}$. The second operator is obtained by formally derivating of operator A_0 with respect to λ

$$A_1 : X_0 \times X_1 \to X_1$$
$$A_1(x,y)(t,\lambda) = \int_t^1 K(t,s)\, x(s,\lambda)\, x(s-t,\lambda)\, ds + \lambda \cdot \left[\int_t^1 K(t,s)\, x(s,\lambda)\, y(s-t,\lambda)\, ds + \int_t^1 K(t,s)\, y(s,\lambda)\, x(s-t,\lambda)\, ds\right]$$

For any $x \in X_0$ and $y_1, y_2 \in X_1$ we have

$$\begin{aligned} \|A_1(x, y_1) - A_1(x, y_2)\| &\leq 2\lambda \cdot I_1 \cdot \|y_1 - y_2\| \leq \\ &\leq \alpha \cdot \|y_1 - y_2\| \end{aligned}$$

which shows that $A_1(x, \cdot)$ is a φ-contraction with $\varphi(t) = \alpha t$. From *Fiber φ-contraction Theorem* we deduce that the operator

$$B_1 : X_0 \times X_1 \to X_0 \times X_1$$
$$B_1 = (A_0, A_1)$$

is a Picard operator and $F_{B_1} = \{(x^*, y^*)\}$, where $\{y^*\} = F_{A_1(x^*,\cdot)}$. Therefore using the same technique as in Theorem 3.1 we prove that $y^* = \frac{\partial x^*}{\partial \lambda}$.

To prove that there exists $\frac{\partial x^*}{\partial t}$ and $\frac{\partial x^*}{\partial t} \in C([0;1] \times [0;\lambda_0])$ we will use the similar way and in this case we consider the operator

$$A_1 : X_0 \times X_1 \to X_1$$
$$A_1(x,y)(t,\lambda) = g'(t) - \lambda K(t,t)\,x(t,\lambda)\,x(0,\lambda) +$$
$$+\lambda \left[\int_t^1 \frac{\partial K}{\partial t}(t,s)\,x(s,\lambda)\,x(s-t,\lambda)\,ds + \int_t^1 K(t,s)\,x(s,\lambda)\,x(s-t,\lambda)\,ds \right].$$

The proof is complete.□

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